

Eigenvalue based analysis and controller synthesis for systems described by delay differential algebraic equations

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Abstract: An eigenvalue based framework is developed for the stability analysis and stabilization of coupled systems with time-delays, which are naturally described by delay differential algebraic equations. The spectral properties of these equations are analyzed and a numerical method for stability assessment is presented, taking into account the effect of small delay perturbations on stability. Subsequently, the design of stabilizing controllers with a pre-scribed structure or order is addressed, based on a direct optimization approach. The effectiveness of the approach is illustrated with numerical examples, and the similarities with the computation and optimization of \mathcal{H}_∞ norms are pointed out.

Keywords: time-delay, differential algebraic equations, stability, fixed order control design

1. INTRODUCTION

We consider the stability analysis and stabilization of systems described by delay differential algebraic equations (DDAEs), also called descriptor systems, of the form

$$E\dot{x}(t) = A_0x(t) + \sum_{i=1}^m A_i x(t - \tau_i), \quad x(t) \in \mathbb{R}^n, \quad (1)$$

where E is allowed to be singular. The time-delays $\tau_i, i = 1, \dots, m$, satisfy

$$0 < \tau_1 < \tau_2 < \dots < \tau_m$$

and the capital letters are real-valued matrices of appropriate dimensions.

The motivation for the system description (1) in the context of designing controllers lies in its generality in modeling interconnected systems. For instance, the feedback interconnection of the system

$$\begin{cases} \dot{z}(t) = \sum F_i z(t - r_i) + \sum G_i u(t - r_i) \\ y(t) = \sum H_i x(t - r_i) + \sum L_i u(t - r_i) \end{cases} \quad (2)$$

and the controller

$$\begin{cases} \dot{z}_c(t) = \sum \hat{F}_i z_c(t - s_i) + \sum \hat{G}_i y(t - s_i) \\ u(t) = \sum \hat{H}_i z_c(t - s_i) + \sum \hat{L}_i y(t - s_i) \end{cases} \quad (3)$$

can be directly brought in the form (1), where

$$x = [z^T \ z_c^T \ u^T \ y^T], \quad \{\tau_1, \dots, \tau_m\} = \{r_i\} \cup \{s_i\}.$$

In this way no elimination of inputs and outputs is required, which may even not be possible in the presence of delays Gumussoy and Michiels [2012]. Another favorable property is the linear dependence of the matrices of the closed-loop system on the elements of the matrices of the controller. The increase in the number of equations, on the contrary, is a minor problem in most applications because the delay difference equations or algebraic constraints are

related to inputs and outputs, as illustrated above, and the number of inputs and outputs is usually much smaller than the number of state variables. Finally, we note that also neutral systems can be dealt with in this framework, by introducing slack variables. The neutral equation

$$\frac{d}{dt} \left(z(t) + \sum_{i=1}^m G_i z(t - \tau_i) \right) = \sum_{i=0}^m H_i z(t - \tau_i) \quad (4)$$

can namely be rewritten as

$$\begin{cases} \dot{v}(t) = \sum_{i=0}^m H_i z(t - \tau_i) \\ 0 = -v(t) + z(t) + \sum_{i=1}^m G_i z(t - \tau_i) \end{cases}, \quad (5)$$

where v is the slack variable. Clearly (5) is of the form (1), if we set $x(t) = [v(t)^T \ z(t)^T]^T$.

The stability analysis of the null solution of (1) in this work is based on a spectrum determined growth property of the solutions, which allows us to infer stability information from the location of the characteristic roots. For instance, exponential stability will be related to a strictly negative spectral abscissa (the supremum of the real parts of the characteristic roots). As we shall see, the spectral abscissa of (1) may not be a continuous function of the delays. Moreover, this may lead to a situation where infinitesimal delay perturbations destabilize an exponentially stable system. These properties are very similar to the spectral properties of neutral equations (see, e.g., [Michiels and Niculescu 2007, Section 2]), which are known to be closely related to DDAEs Fridman and Shaked [2002]. Since in a practical control design the robustness of stability against infinitesimal changes of parameters is a prerequisite, we will define the concept of strong stability, inspired by the common terminology in the context of neutral equations Hale and Verduyn Lunel [2002], and we will

introduce the notion of the robust spectral abscissa, which explicitly takes into account the effect of small parametric perturbations. We will also provide explicit conditions and expressions that eventually lead to numerical algorithms.

Numerical algorithms for the computation of characteristic roots and the robust spectral abscissa are outlined, and subsequently applied to the design of stabilizing controllers. Similarly to Vanbiervliet et al. [2008], a direct optimization approach towards stabilization is taken, based on minimizing the (robust) spectral abscissa as a function of the parameters of the controller. In the example (2)-(3) these parameters may correspond to elements of the controller matrices. In this way stabilization is achieved on the moment that the objective function becomes strictly negative. This approach allows us to design stabilizing controllers with a prescribed structure or order (dimension). It is also possible to fix elements of the controller matrices, allowing to impose additional structure, e.g., a proportional-integral-derivative (PID)-like structure, or sparsity.

In the context of stability optimization of linear time-invariant (LTI) systems it is well known that the spectral abscissa is in general a nonconvex function of the elements of the system matrices. In addition, it is typically not everywhere differentiable, even not everywhere Lipschitz continuous, although it is differentiable almost everywhere Burke et al. [2002], Gumussoy and Overton [2008]. These properties carry over to the case of the robust spectral abscissa of DDAEs under consideration. Therefore, special optimization methods for nonsmooth problems will be used, more precisely, the same methods underlying the package HIFOO for fixed-order \mathcal{H}_∞ control design for LTI systems Gumussoy and Overton [2008]. The main differences with the stabilization routine in HIFOO are i) the *infinite*-dimensional plant is controlled by a finite-dimensional controller, and ii) the potential sensitivity of the spectral abscissa w.r.t. infinitesimal perturbations of system parameters (delays), which much be taken into account.

At the end of the paper we point out how the computational and optimization of \mathcal{H}_∞ norm leads to similar problems as well as similar solutions and algorithms.

2. PRELIMINARIES AND ASSUMPTIONS

Let matrix E in (1) satisfy

$$\text{rank}(E) = n - \nu,$$

with $1 \leq \nu < n$, and let the columns of matrix $U \in \mathbb{R}^{n \times \nu}$, respectively $V \in \mathbb{R}^{n \times \nu}$, be a (minimal) basis for the right, respectively left nullspace of E , which implies

$$U^T E = 0, \quad EV = 0. \quad (6)$$

Throughout the paper we make the following assumption.

Assumption 2.1. The matrix $U^T A_0 V$ is nonsingular.

The equations (1) can be separated into coupled delay differential and delay difference equations. When we define

$$\mathbf{U} = [U^\perp \ U], \quad \mathbf{V} = [V^\perp \ V],$$

a pre-multiplication of (1) with \mathbf{U}^T and the substitution

$$x = \mathbf{V} [x_1^T \ x_2^T]^T,$$

with $x_1(t) \in \mathbb{R}^{n-\nu}$ and $x_2(t) \in \mathbb{R}^\nu$, yield the coupled equations

$$\begin{aligned} E^{(11)} \dot{x}_1(t) &= \sum_{i=0}^m A_i^{(11)} x_1(t - \tau_i) + \sum_{i=0}^m A_i^{(12)} x_2(t - \tau_i), \\ 0 &= A_0^{(22)} x_2(t) + \sum_{i=1}^m A_i^{(22)} x_2(t - \tau_i) + \sum_{i=0}^m A_i^{(21)} x_1(t - \tau_i), \end{aligned} \quad (7)$$

where

$$E^{(11)} = U^\perp{}^T E V^\perp \quad (8)$$

and

$$\begin{aligned} A_i^{(11)} &= U^\perp{}^T A_i V^\perp, \quad A_i^{(12)} = U^\perp{}^T A_i V, \\ A_i^{(21)} &= U^T A_i V^\perp, \quad A_i^{(22)} = U^T A_i V, \quad i = 0, \dots, m. \end{aligned} \quad (9)$$

Matrix $E^{(11)}$ is invertible, following from

$$\text{rank}(E^{(11)}) = \text{rank}(U^T E V) = \text{rank}(E) = n - \nu.$$

In addition, Assumption 2.1 corresponds to the invertibility of matrix $A_0^{(22)}$.

Equations (7) are semi-explicit delay differential algebraic equations of index 1, because delay differential equations are obtained by differentiating the second equation. It precludes the occurrence of impulsive solutions Fridman and Shaked [2002]. Moreover, the invertibility of $A_0^{(22)}$ prevents that the equations are of advanced type and, hence, non-causal.

3. SPECTRAL PROPERTIES AND STABILITY

In this section the spectral properties of equation (1) are discussed. In the technical derivation connections with the neutral equation obtained by differentiating the second equation in (7), play an important role.

3.1 Exponential stability

Stability conditions for the zero solution of (1) can be expressed in terms of the position of the *characteristic roots*, i.e., the roots of the equation

$$\det \Delta(\lambda) = 0, \quad (10)$$

where Δ is the characteristic matrix,

$$\Delta(\lambda) := \lambda E - A_0 - \sum_{i=1}^m A_i e^{-\lambda \tau_i}.$$

In particular, we have the following result.

Proposition 3.1. The null solution of (1) is exponentially stable if and only if $c < 0$, where c is the *spectral abscissa*, $c := \sup \{\Re(\lambda) : \det \Delta(\lambda) = 0\}$.

3.2 Continuity of the spectral abscissa and strong stability

We discuss the dependence of the spectral abscissa of (1) on the delay parameters $\boldsymbol{\tau} = (\tau_1, \dots, \tau_m)$. In general the function

$$\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m \mapsto c(\boldsymbol{\tau}) \quad (11)$$

is not everywhere continuous, which carries over from the spectral properties of delay difference equations (see, e.g., Avellar and Hale [1980], Michiels et al. [2002, 2009]). In the light of this we first outline properties of the function

$$\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m \mapsto c_D(\boldsymbol{\tau}) := \sup \{\Re(\lambda) : \det \Delta_D(\lambda; \boldsymbol{\tau}) = 0\}, \quad (12)$$

with

$$\Delta_D(\lambda; \boldsymbol{\tau}) := U^T A_0 V + \sum_{i=1}^m U^T A_i V e^{-\lambda \tau_i}. \quad (13)$$

Note that (13) can be interpreted as the characteristic matrix of the delay difference equation

$$U^T A_0 V z(t) + \sum_{i=1}^m U^T A_i V z(t - \tau_i) = 0, \quad (14)$$

associated with the neutral equation obtained by differentiating the second equation in (7).

The property that the function (12) is not continuous led in Michiels and Vyhldal [2005] to the smallest upper bound, which is ‘insensitive’ to small delay changes.

Definition 3.2. For $\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m$, let $C_D(\boldsymbol{\tau}) \in \mathbb{R}$ be defined as

$$C_D(\boldsymbol{\tau}) := \lim_{\epsilon \rightarrow 0^+} c_D^\epsilon(\boldsymbol{\tau}),$$

where

$$c_D^\epsilon(\boldsymbol{\tau}) := \sup \{c_D(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) : \delta\boldsymbol{\tau} \in \mathbb{R}^m \text{ and } \|\delta\boldsymbol{\tau}\| \leq \epsilon\}.$$

Several properties of this upper bound on c_D are listed below (see [Michiels and Niculescu 2007, Chapter 2] for an overview).

Proposition 3.3. The following assertions hold:

(1) the function

$$\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m \mapsto C_D(\boldsymbol{\tau})$$

is continuous;

(2) for every $\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m$, the quantity $C_D(\boldsymbol{\tau})$ is equal to the unique zero of the strictly decreasing function

$$\zeta \in \mathbb{R} \rightarrow f(\zeta; \boldsymbol{\tau}) - 1, \quad (15)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is defined by

$$f(\zeta; \boldsymbol{\tau}) := \max_{\boldsymbol{\theta} \in [0, 2\pi]^m} \rho \left(\sum_{k=1}^m (U^T A_0 V)^{-1} (U^T A_k V) e^{-\zeta \tau_k} e^{j\theta_k} \right); \quad (16)$$

(3) $C_D(\boldsymbol{\tau}) = c_D(\boldsymbol{\tau})$ for rationally independent¹ $\boldsymbol{\tau}$;

(4) for all $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in (\mathbb{R}_0^+)^m$, we have

$$\text{sign}(C_D(\boldsymbol{\tau}_1)) = \text{sign}(C_D(\boldsymbol{\tau}_2)) := \Xi; \quad (17)$$

(5) $\Xi < 0$ (> 0) holds if and only if $\gamma_0 < 1$ (> 1) holds, where

$$\gamma_0 := \max_{\boldsymbol{\theta} \in [0, 2\pi]^m} \rho \left(\sum_{k=1}^m (U^T A_0 V)^{-1} (U^T A_k V) e^{j\theta_k} \right). \quad (18)$$

For the single delay case, some of the expressions can be simplified.

Corollary 3.4. If $m = 1$ then we have

$$C_D(\boldsymbol{\tau}) = \frac{1}{\tau_1} \log \left\{ \rho \left((U^T A_0 V)^{-1} (U^T A_1 V) \right) \right\}$$

and

$$\gamma_0 = \rho \left((U^T A_0 V)^{-1} (U^T A_1 V) \right).$$

¹ The m components of $\boldsymbol{\tau} = (\tau_1, \dots, \tau_m)$ are rationally independent if and only if $\sum_{k=1}^m n_k \tau_k = 0$, $n_k \in \mathbb{Z}$ implies $n_k = 0$, $\forall k = 1, \dots, m$. For instance, two delays τ_1 and τ_2 are rationally independent if their ratio is an irrational number.

We now come back to the DDAE (1), more precisely, to the properties of the spectral abscissa function (11). The following two technical lemmas make connections between the characteristic roots of (1) and the zeros of (13). Their proofs are similar to the proofs of the corresponding results for neutral equations, in particular [Michiels and Niculescu 2007, Propositions 1.26 and 1.27].

Lemma 3.5. There exists a sequence $\{\lambda_k\}_{k \geq 1}$ of characteristic roots of (1) satisfying

$$\lim_{k \rightarrow \infty} \Re(\lambda_k) = c_D, \quad \lim_{k \rightarrow \infty} \Im(\lambda_k) = \infty.$$

Lemma 3.6. For every $\epsilon > 0$ there exists a continuous function $R : (\mathbb{R}_0^+)^m \rightarrow \mathbb{R}^+$, such the characteristic roots of (1) in the half plane

$$\{\lambda \in \mathbb{C} : \Re(\lambda) \geq C_D(\boldsymbol{\tau}) + \epsilon\}. \quad (19)$$

have modulus smaller than $R(\boldsymbol{\tau})$, for all $\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m$.

The lack of continuity of the spectral abscissa function (11) leads us again to an upper bound that takes into account the effect of small delay perturbations.

Definition 3.7. For $\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m$, let the *robust spectral abscissa* $C(\boldsymbol{\tau})$ be defined as

$$C(\boldsymbol{\tau}) := \lim_{\epsilon \rightarrow 0^+} c^\epsilon(\boldsymbol{\tau}), \quad (20)$$

where

$$c^\epsilon(\boldsymbol{\tau}) := \sup \{c(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) : \delta\boldsymbol{\tau} \in \mathbb{R}^m \text{ and } \|\delta\boldsymbol{\tau}\| \leq \epsilon\}.$$

The following characterization of the robust spectral abscissa (20) constitutes the main result of this section. Its proof can be found in Michiels [2010a].

Proposition 3.8. The following assertions hold:

(1) the function

$$\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m \mapsto C(\boldsymbol{\tau}) \quad (21)$$

is continuous;

(2) for every $\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m$, we have

$$C(\boldsymbol{\tau}) = \max(C_D(\boldsymbol{\tau}), c(\boldsymbol{\tau})). \quad (22)$$

In line with the sensitivity of the spectral abscissa with respect to infinitesimal delay perturbations, which has been resolved by considering the robust spectral abscissa (20) instead, we define the concept of strong stability².

Definition 3.9. The null solution of (1) is strongly exponentially stable if there exists a number $\hat{\tau} > 0$ such that the null solution of

$$E\dot{x}(t) = A_0 + \sum_{k=1}^m A_k x(t - (\tau_k + \delta\tau_k))$$

is exponentially stable for all $\delta\boldsymbol{\tau} \in (\mathbb{R}^+)^m$ satisfying $\|\delta\boldsymbol{\tau}\| < \hat{\tau}$ and $\tau_k + \delta\tau_k \geq 0$, $k = 1, \dots, m$.

The following result provides necessary and sufficient conditions.

Theorem 3.10. The null solution of (1) is strongly exponentially stable if and only if $C(\boldsymbol{\tau}) < 0$, or, equivalently, $c(\boldsymbol{\tau}) < 0$ and $\gamma_0 < 1$, where γ_0 is defined by (18).

² This terminology is borrowed from the theory of neutral delay differential equations Hale and Verduyn Lunel [2002], Michiels and Vyhldal [2005].

4. STABILITY ASSESSMENT

4.1 Computation of characteristic roots

The characteristic roots are the solutions of the nonlinear eigenvalue problem

$$\Delta(\lambda)v = 0, \quad v \in \mathbb{C}^n, \quad v \neq 0. \quad (23)$$

It can be verified that they also correspond to the eigenvalues of the linear operator \mathcal{A} on the space X , defined by

$$\mathcal{D}(\mathcal{A}) = \left\{ \phi \in X : \begin{aligned} \phi' &\in \mathcal{C}([-\tau_m, 0], \mathbb{R}^n), \\ E\phi'(0) &= A_0\phi(0) + \sum_{k=1}^m A_k\phi(-\tau_k) \end{aligned} \right\}, \quad (24)$$

$$\mathcal{A}\phi = \phi',$$

whose eigenvalue problem is given by

$$(\lambda I - \mathcal{A})z = 0, \quad z \in \mathcal{D}(\mathcal{A}), \quad z \neq 0. \quad (25)$$

Our method for computing characteristic roots exploits the two representations, and can be sketched as follows.

- Algorithm 4.1.* (1) Fix N and compute the eigenvalues of the pencil (E_N, A_N) , obtained by a spectral discretization of the linear operator \mathcal{A} .
(2) Correct the approximate characteristic roots by applying Newton's method to the nonlinear equations (23).

For more details we refer to, e.g., Michiels [2010a], Gumusoy and Michiels [2012].

4.2 Computation of the robust spectral abscissa

Assessing the growth rate of solutions and stability cannot always be reduced to computing the characteristic roots in a compact set (note that a spectral discretization of \mathcal{A} corresponds in the frequency domain to a rational approximation around the origin), since system (1) may have infinite series of characteristic roots whose imaginary parts tend to infinity, yet whose real parts have a finite limit (see, e.g., Lemma 3.5). Moreover, the presence of such root chains may lead to a discontinuity of the spectral abscissa function with respect to the delays. The latter can be resolved by considering the robust spectral abscissa (20).

Based on the characterization (22) and Lemma 3.6 the robust spectral abscissa can be calculated by complementing the computation of characteristic roots, with the evaluation of C_D . The latter is outlined in what follows.

We take a predictor-corrector approach to compute C_D , based on the second assertion of Proposition 3.3. In the prediction step we use Dekker-Brent's method Atkinson [1989] to find a zero of the function (15), where the function evaluations of f are approximated by restricting θ in (16) to a grid. Note that for $m \geq 2$ we can write

$$f(\zeta; \tau) = \max_{\theta \in [0, 2\pi]^{m-1}} \rho \left(H_1 e^{-\zeta\tau_1} + \sum_{k=2}^m H_k e^{-\zeta\tau_k} e^{j\theta_k} \right), \quad (26)$$

with

$$H_k = (U^T A_0 V)^{-1} (U^T A_k V), \quad k = 1, \dots, m,$$

hence, a grid on the space $[0, 2\pi]^{m-1}$ is sufficient. If a high accuracy of C_D is required, then one may want to use a local corrector, based on the equations

$$\begin{cases} \left(H_1 e^{-\zeta\tau_1} + \sum_{k=2}^m H_k e^{-\zeta\tau_k} e^{j\theta_k} \right) v = \lambda v \\ u^* \left(H_1 e^{-\zeta\tau_1} + \sum_{k=2}^m H_k e^{-\zeta\tau_k} e^{j\theta_k} \right) = \lambda u^* \\ n(u) = 1 \\ u^* v = 1 \\ \lambda^* \lambda = 1 \\ \Im \left(e^{-\zeta\tau_k} e^{j\theta_k} (u^* H_k v) \bar{\lambda} \right) = 0, \quad k = 2, \dots, m \end{cases}, \quad (27)$$

where $n(u) = 1$ is a normalization constraint. These equations express that for the desired value of ζ , the matrix

$$H_1 e^{-\zeta\tau_1} + \sum_{k=2}^m H_k e^{-\zeta\tau_k} e^{j\theta_k}$$

has an eigenvalue on the unit circle and that the derivatives of the modulus of this eigenvalue with respect to $\theta_2, \dots, \theta_m$ are equal to zero. Since the (overdetermined) equations have an exact solution, the Gauss-Newton method exhibits quadratic convergence whenever the solution is isolated, see Section 10.2 of Nocedal and Wright [1999].

Remark 4.2. To assess strong stability of the null solution of (1), it is not necessary to compute C_D , in addition to computing rightmost characteristic roots. By Theorem 3.10 it is only needed to check whether $C_D < 0$, which is equivalent to $\gamma_0 < 1$. Because

$$\gamma_0 = f(0; \tau),$$

this amounts to evaluating the function f in one point, instead of finding the zero of (15). The value of γ_0 can be extracted from the solutions of (26) and (27), where ζ is set to zero and the equation $\lambda^* \lambda = 1$ in (27) is dropped.

Remark 4.3. The computational cost of C_D is dominated by the evaluation of the right hand side of (26), where gridding leads to an exponential growth in the number of terms and restricts the approach to a small number of delays. However, in most practical problems, the number of delays to be considered in (26) is much smaller than the number of system delays, m , because most of the terms in (26) are zero. Note that in the context of feedback control a nonzero term corresponds to a high frequency feedthrough over the whole control loop.

5. ROBUST STABILIZATION BY EIGENVALUE OPTIMIZATION

We now consider the equations

$$E\dot{x}(t) = A_0(\mathbf{p})x(t) + \sum_{i=1}^m A_i(\mathbf{p})x(t - \tau_i), \quad (28)$$

where the system matrices linearly depend on parameters $\mathbf{p} \in \mathbb{R}^{n_p}$. In control applications these parameter usually correspond to controller parameters. For example, in the feedback interconnection (2)-(3) they may arise from a parameterization of the matrices $(\hat{F}_i, \hat{G}_i, \hat{H}_i, \hat{L}_i)$. In §5.1 the stabilization problem for (28) is related to two optimization problems and in §5.2 the corresponding optimization algorithms are briefly discussed.

5.1 An optimization point of view

To impose exponential stability of the null solution of (28) it is necessary to find values of \mathbf{p} for which the spectral abscissa is strictly negative. If the achieved stability is required to be robust against small delay perturbations, this requirement must be strengthened to the negativeness of the robust spectral abscissa. This brings us to the optimization problem

$$\inf_{\mathbf{p}} C(\boldsymbol{\tau}; \mathbf{p}). \quad (29)$$

Strongly stabilizing values of \mathbf{p} exist if the objective function can be made strictly negative. By Theorem 3.10 the latter can be evaluated as

$$C_D(\boldsymbol{\tau}; \mathbf{p}) = \max(c(\boldsymbol{\tau}; \mathbf{p}), C_D(\boldsymbol{\tau}; \mathbf{p})). \quad (30)$$

An alternative approach consists of solving the constrained optimization problem

$$\inf_{\mathbf{p}} c(\boldsymbol{\tau}; \mathbf{p}), \text{ subject to } \gamma_0(\mathbf{p}) < \gamma, \quad (31)$$

where $\gamma < 1$. If the objective function is strictly negative, then the satisfaction of the constraint implies strong stability.

The advantage of solving problem (29) is that for all values of \mathbf{p} a bound on the exponential growth rate of the solutions is assured, which takes into account delay perturbations. The advantage of solving (31) instead is that evaluating $\gamma_0(\mathbf{p})$ is less computationally demanding than evaluating $C_D(\boldsymbol{\tau}; \mathbf{p})$, see Remark 4.2.

5.2 Algorithms

We start with problem (29). Lemma 3.6 illustrates that the spectrum of the DDAE (28) behaves similarly as the spectrum of a delay differential equation of retarded type whenever the parameters are restricted to the set

$$\{\mathbf{p} \in \mathbb{R}^{n_p} : c(\boldsymbol{\tau}; \mathbf{p}) > C_D(\boldsymbol{\tau}; \mathbf{p})\}. \quad (32)$$

Also the spectral properties for retarded systems derived in Vanbiervliet et al. [2008] carry over whenever (32) is satisfied. In particular, the spectral abscissa function $\mathbf{p} \mapsto c(\boldsymbol{\tau}; \mathbf{p})$ may be not everywhere differentiable, even not everywhere Lipschitz continuous. A lack of differentiability may occur when there is more than one *active* characteristic root, i.e., a characteristic root whose real part equals the spectral abscissa. A lack of Lipschitz continuity may occur when an active characteristic roots is multiple and non-semisimple. On the contrary, the spectral abscissa function is differentiable at points where there is only one active characteristic root with multiplicity one. Since this is the case with probability one when randomly sampling parameters from the set (32), the spectral abscissa is smooth almost everywhere. The function f in Proposition 3.3 is a maximum eigenvalue function, similarly to the spectral abscissa function. Therefore, the above properties also hold for the function

$$\mathbf{p} \mapsto C_D(\boldsymbol{\tau}; \mathbf{p})$$

and they are preserved by the maximum operator in (30). We conclude that the objective function in (29), the robust spectral abscissa, is not everywhere differentiable, not everywhere Lipschitz continuous, but it is differentiable almost everywhere.

The properties of the problem (29) preclude the use of standard optimization methods, developed for smooth

problems. Instead we use a combination of BFGS with weak Wolfe line search and gradient sampling, as implemented in the MATLAB code HANSO Overton [2009]. The overall algorithm only requires the evaluation of the objective function, as well as its derivatives with respect to the controller parameters, *whenever* it is differentiable.

The evaluation of the robust spectral abscissa can be done as described in §4.2. When this function is differentiable we can express

$$\frac{\partial C}{\partial p_k}(\boldsymbol{\tau}; \mathbf{p}) = \begin{cases} \frac{\partial c}{\partial p_k}(\boldsymbol{\tau}; \mathbf{p}), & c(\boldsymbol{\tau}; \mathbf{p}) > C_D(\boldsymbol{\tau}; \mathbf{p}), \\ \frac{\partial C_D}{\partial p_k}(\boldsymbol{\tau}; \mathbf{p}), & c(\boldsymbol{\tau}; \mathbf{p}) < C_D(\boldsymbol{\tau}; \mathbf{p}), \end{cases}$$

for $k = 1, \dots, n_p$. In case $c > C_D$, the derivative of the robust spectral abscissa is inferred from the sensitivity of an individual characteristic roots. More precisely, we can express

$$\frac{\partial C}{\partial p_k}(\boldsymbol{\tau}; \mathbf{p}) = \Re \left(\frac{w^* \left(\frac{\partial A_0}{\partial p_k}(\mathbf{p}) + \sum_{i=1}^m \frac{\partial A_i}{\partial p_k}(\mathbf{p}) e^{-\lambda \tau_i} \right) z}{w^* \left(E + \sum_{i=1}^m A_i \tau_i e^{-\lambda \tau_i} \right) z} \right), \quad (33)$$

where the tuple (λ, w, z) satisfies

$$w^* \Delta(\lambda) = 0, \quad \Delta(\lambda)z = 0, \quad z \neq 0, \quad w \neq 0$$

and λ corresponds to the rightmost characteristic roots. If $c < C_D$ then the robust spectral abscissa is differentiable in the generic case where for $\zeta = C_D(\boldsymbol{\tau}; \mathbf{p})$ the maximum in (26) is isolated. We can then express

$$\frac{\partial C}{\partial p_k}(\boldsymbol{\tau}; \mathbf{p}) = \frac{\Re \left(\bar{\lambda} u^* \left(\frac{\partial H_1}{\partial p_k}(\mathbf{p}) e^{-\zeta \tau_1} + \sum_{i=2}^m \frac{\partial H_i}{\partial p_k}(\mathbf{p}) e^{-\zeta \tau_i} e^{j\theta_i} \right) v \right)}{\Re \left(\bar{\lambda} u^* \left(H_1(\mathbf{p}) \tau_1 e^{-\zeta \tau_1} + \sum_{i=2}^m H_i(\mathbf{p}) \tau_i e^{-\zeta \tau_i} e^{j\theta_i} \right) v \right)}, \quad (34)$$

where $(\zeta, \lambda, u, v, \boldsymbol{\theta})$ refers to the corresponding solution of (27). An alternative to the analytic formulae (33) and (34) consists of computing the derivatives by finite differences.

Finally, we come back to the constrained problem (31). It can be solved using the barrier method proposed in Vyhldal et al. [2010], which is on its turn inspired by interior point methods for solving convex optimization problems, see, e.g., Boyd and Vandenberghe [2004]. The first step consists of finding a feasible point, i.e., a set of values for \mathbf{p} satisfying the constraint. If the feasible set is nonempty such a point can be found by solving

$$\min_{\mathbf{p}} \gamma_0(\mathbf{p}). \quad (35)$$

Once a feasible point $\mathbf{p} = \mathbf{p}_0$ has been obtained one can solve in the next step the unconstrained optimization problem

$$\min_{\mathbf{p}} \{c(\mathbf{p}) - r \log(\gamma - \gamma_0(\mathbf{p}))\} \quad (36)$$

where $r > 0$ is a small number and γ satisfies

$$\gamma_0(\mathbf{p}) < \gamma \leq 1.$$

The second term (the barrier) assures that the feasible set cannot be left when the objective function is decreased in a quasi-continuous way (because the objective function will go to infinity when $\gamma_0 \rightarrow \gamma$). If (36), with $\gamma = 1$, is repeatedly solved for decreasing values of r and with the previous solution as starting value, a solution of (31) is obtained. Strong *exponential* stability can be imposed by setting γ strictly smaller than one.

6. NUMERICAL EXAMPLES

We illustrate the approach with several case-studies. A user-friendly MATLAB implementation of the control design algorithms, as well as all data corresponding to the presented examples, are publicly available Michiels [2010b].

6.1 A stabilization problem with input delay

As a first example we take the system with input delay from Vanbiervliet et al. [2008]:

$$\dot{x}(t) = Ax(t) + Bu(t - \tau), \quad y(t) = x(t), \quad (37)$$

where

$$A = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \quad \tau = 5. \quad (38)$$

The uncontrolled system is unstable, characterized by the spectral abscissa $c = 0.108$. We design a dynamic controller of the form

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c y(t), \\ u(t) = C_c x_c(t) + D_c y(t), \end{cases} \quad x_c(t) \in \mathbb{R}^{n_c}, \quad (39)$$

using the approach of Section 5, where we set

$$p = \text{vec} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}.$$

Since the transfer function from u to y is strictly proper, the robust spectral abscissa equals the spectral abscissa, and the optimization problems (29) and (31) reduce to the (unconstrained) minimization of the spectral abscissa. The results for different controller orders are displayed in Table 1. The case $n_c = 0$ refers to static feedback, also considered in Vanbiervliet et al. [2008]. In Figure 1 the rightmost characteristic roots of the closed-loop system are shown for $n_c = 0$ and $n_c = 3$.

controller	minimum c
output feedback, $n_c = 0$	-0.1489
output feedback, $n_c = 1$	-0.2293
output feedback, $n_c = 2$	-0.2682
output feedback, $n_c = 3$	-0.4575

Table 1. Results of minimizing the spectral abscissa of (37) and (39).

For the second example we assume that the measured output of the system (37) is instead given by

$$\tilde{y}(t) = x(t) + \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} u(t - 2.5) + \begin{bmatrix} 2/5 \\ -2/5 \\ -2/5 \end{bmatrix} u(t - 5) \quad (40)$$

and we design a static controller,

$$u(t) = D_c \tilde{y}(t). \quad (41)$$

In this case there is a high-frequency feedthrough term in the control loop. Solving the optimization problem (29) leads to

$$C = -0.0309, \quad D_c = [0.0409 \ 0.0612 \ 0.3837]. \quad (42)$$

In Figure 2 we show the rightmost characteristic roots corresponding to the minimum of the robust spectral abscissa (42). The dotted line corresponds to $\Re(\lambda) = c_D$, the dashed line to $\Re(\lambda) = C_D$. The minimum of C is characterized by an equality between C_D and the spectral

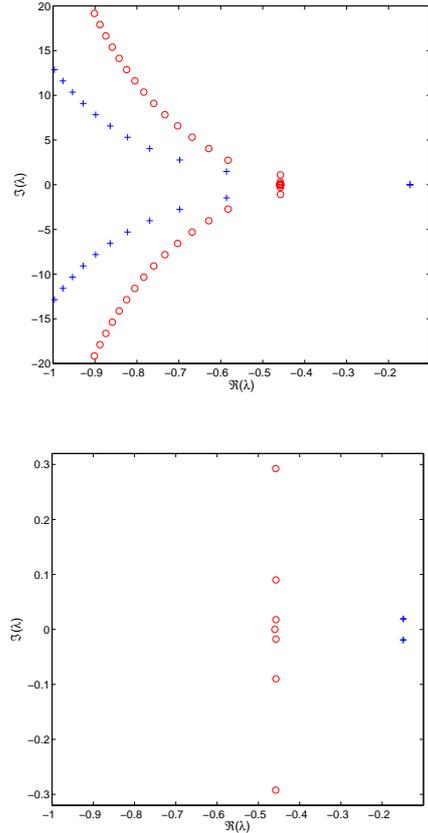


Fig. 1. Characteristic roots of (37) and (39), corresponding to a minimum of the spectral abscissa, for $n_c = 0$ (pluses) and $n_c = 3$ (circles). The top and bottom pane correspond to a different scaling of the imaginary axis.

abscissa c , the latter induced by a rightmost characteristic root with multiplicity three. This is compatible with the three degrees of freedom in the controller. In order to illustrate that we indeed have $c = C_D$ we depict in Figure 3 the rightmost characteristic roots after perturbing the delay value 2.5 in (40) to 2.51.

Solving the optimization problem (36) with $r = 10^{-3}$ and $\gamma = 10^{-3}$ yields

$$c = -0.0345, \quad C_D = -0.00602, \quad D_c = [0.0249 \ 0.1076 \ 0.3173].$$

Compared to (42), where we had $C = c = C_D$, a further reduction of the spectral abscissa has been achieved, at the price of an increased value of C_D . This is expected because the constraint $\gamma_0 < 1$ imposes robustness of stability, yet no bound on the exponential decay rate of the solutions.

6.2 Heating system

In Vyhldal et al. [2009] a linear model of an experimental heat transfer set-up at the CTU in Prague is proposed, consisting of 10 delay differential equations. The inclusion of an integrator to achieve a zero steady state error to a set-point of one of the state variables eventually leads to equations of the form

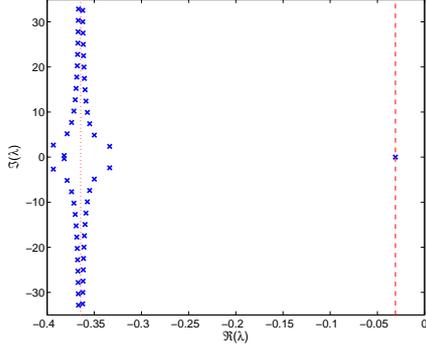


Fig. 2. Characteristic roots corresponding to the minimum of the robust spectral abscissa of (37) and (40), for the control law (41). The rightmost characteristic roots, $\lambda \approx -0.0309$, has multiplicity three.

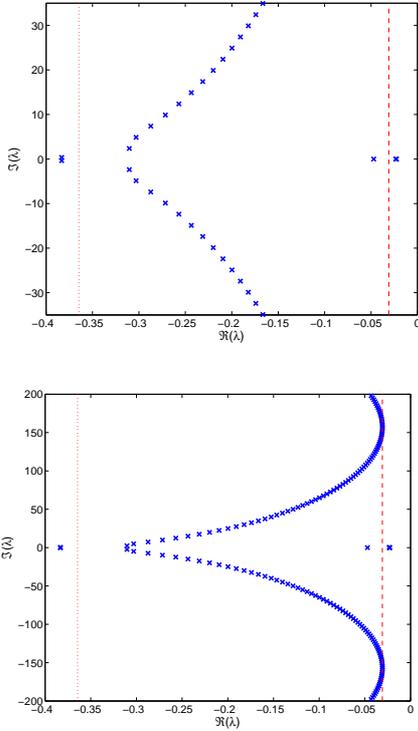


Fig. 3. Effect on the characteristic roots of a perturbation of the delays (2.5, 5) in (40) to (2.51, 5). The difference between top and bottom pane lies in the scaling of the imaginary axis.

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^5 A_i x(t - \tau_i) + Bu(t - \tau_6), \quad (43)$$

$$x(t) \in \mathbb{R}^{11 \times 11}, \quad u(t) \in \mathbb{R},$$

see Vyhřídál et al. [2009] for the corresponding matrices and delay values. We consider two outputs

$$y(t) = \begin{bmatrix} x_5(t) + x_6(t) \\ -x_{10}(t) + x_{11}(t) \end{bmatrix}. \quad (44)$$

The spectral abscissa of the uncontrolled system is equal to zero. In Table 2 we show the result of optimizing

the spectral abscissa using static state feedback, $u(t) = Kx(t)$, $K \in \mathbb{R}^{1 \times 11}$, and dynamic output feedback (39).

controller	minimum c
static state feedback,	-0.0577
output feedback, $n_c = 0$	-0.0187
output feedback, $n_c = 1$	-0.0218
output feedback, $n_c = 2$	-0.0237

Table 2. Results of minimizing the spectral abscissa of (43)-(44) for static state feedback and for dynamic output feedback (39).

7. DUALITY WITH THE \mathcal{H}_∞ PROBLEM

In a practical control design the stabilization phase is usually only a first step in the overall design procedure. Consider now the (subsequent) fixed-order \mathcal{H}_∞ synthesis problem, where the aim is to optimize the \mathcal{H}_∞ norm of

$$G(\lambda) = C \left(\lambda E - A_0 - \sum_{i=1}^m A_i e^{-\lambda \tau_i} \right)^{-1} B$$

as a function of parameters on which the system matrices depend.

It turns out the function $\tau \mapsto \|G(j\omega; \tau)\|_{\mathcal{H}_\infty}$ has a very similar behavior to the spectral abscissa function (11). In particular it is not everywhere continuous. Moreover, the discontinuities are all related to the behavior of the transfer function at large frequencies (analogous to the behavior of eigenvalues with large imaginary parts in §3.2). This high frequency behavior is described by the associated *asymptotic transfer function*

$$G_a(\lambda) := -CV \left(U^T A_0 V + \sum_{i=1}^m U^T A_i V e^{-\lambda \tau_i} \right)^{-1} U^T B,$$

which takes the role of the associated delay-difference equation (14). Finally, the sensitivity w.r.t. small delay perturbations leads to the definition of the *strong \mathcal{H}_∞ norm* (analogous to strong stability), defined as:

$$\| \| G(j\omega; \tau) \| \|_{\mathcal{H}_\infty} := \lim_{\epsilon \rightarrow 0^+} \sup \{ \| G(j\omega; \tau + \delta\tau) \|_{\mathcal{H}_\infty} : \delta\tau \in (\mathbb{R}^+)^m, \| \delta\tau \|_2 < \epsilon \}.$$

The computation of the strong \mathcal{H}_∞ norm involves a tradeoff between the behavior of the transfer function at small and large frequencies, similar to the result of Theorem 3.10 on strong stability, and it can be optimized using the same algorithms.

For more details, we refer to Gumussoy and Michiels [2012].

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