Computing the Pseudospectral Abscissa of Time-Delay Systems

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Abstract: The pseudospectra of a linear time-invariant system are the sets in the complex plane consisting of all the roots of the characteristic equation when the system matrices are subjected to all possible perturbations with a given upper bound. The pseudospectral abscissa are defined as the maximum real part of the characteristic roots in the pseudospectra and, therefore, they are for instance important from a robust stability point of view. In this paper we present a numerical method for the computation of the pseudospectral abscissa of retarded delay differential equations with discrete pointwise delays. Our approach is based on the connections between the pseudospectra and the level sets of an appropriately defined complex function. These connections lead us to a bisection algorithm for the computation of the pseudospectral abscissa, where each step relies on checking the presence of imaginary axis eigenvalues of an appropriately defined operator. Because this operator is infinite-dimensional a predictor-corrector approach is taken. In the predictor step the bisection algorithm is applied where the operator is discretized into a matrix, yielding approximations for the pseudospectral abscissa. The effect of the discretization is fully characterized in the paper. In the corrector step, the approximate pseudospectral abscissa are corrected to any given accuracy, by solving a set of nonlinear equations that characterize extreme points in the pseudospectra contours.

Keywords: pseudospectra, pseudospectral abscissa, computational methods, time-delay, delay equations, robustness, stability.

1. INTRODUCTION

The pseudospectra provide information about the characteristic roots of the system when the system matrices in the characteristic equation are subject to perturbations. They are closely related with the robust stability of the system and distance to instability, Trefethen (1997). We consider the characteristic equation of the time-delay systems:

$$\det F(\lambda) = 0 \tag{1}$$

where

$$F(\lambda) := \lambda I_{n\times n} - \left(\sum_{i=0}^{m} A_i e^{-\lambda\tau_i}\right), \tag{2}$$

$A_i \in \mathbb{C}^{n\times n}$, $\tau_i \in \mathbb{R}_0^+$ for $i = 0,\ldots,m$ and $\tau_0 = 0$. The maximum real part of the characteristic roots is the *spectral abscissa*,

$$\alpha_0 = \sup_{\lambda\in\mathbb{C}}\{\Re(\lambda) : \det F(\lambda) = 0\}. \tag{3}$$

When the system matrices in (2) is subject to the perturbations, the *pseudospectra* of the characteristic equation (1) is defined as

$$\Lambda_\epsilon = \{\lambda \in \mathbb{C} : \det\left(F(\lambda) + \Delta F(\lambda)\right) = 0\} \tag{4}$$

where perturbations on the systems matrices are represented as

$$\Delta F(\lambda) := -\left(\sum_{i=0}^{m} \delta A_i e^{-\lambda\tau_i}\right), \tag{5}$$

$\delta A_i \in \mathbb{C}^{n\times n}$ and satisfying $\|\delta A_i\|_2 \le \frac{\epsilon}{w_i}$ for $i = 0,\ldots,m$. Here the $w_i \in \mathbb{R}_0^+$ $i = 0,\ldots,m$ are some weights on the perturbations which can be chosen apriori. The maximum real part in the pseudospectra is the *pseudospectral abscissa* which is defined as

$$\alpha_\epsilon = \sup_{\lambda\in\mathbb{C}}\{\Re(\lambda) : \lambda \in \Lambda_\epsilon(F)\}. \tag{6}$$

The computation of the pseudospectral abscissa for finite dimensional systems corresponds to the special case of (1):

$$F_0(\lambda) = \lambda I_n - A_0.$$

In this particular case, the pseudospectra can be equivalently expressed as Boyd et al. (1989)

$$\Lambda_\epsilon^0 = \left\{\lambda \in \mathbb{C} : \sigma_{\max}\left(F_0^{-1}(\lambda)\right) > \frac{1}{\epsilon}\right\} \tag{7}$$

where $\sigma_{\max}(A)$ is the largest singular value of the matrix A. Note that this definition reduces the pseudospectra boundary to the level set of the resolvent norm. This connection is used to compute the distance to instability and pseudospectral abscissa via a bisection algorithm in Byers (1988) and Burke et al. (2003a) respectively. A quadratically convergent algorithm for the pseudospectral abscissa computation is given in Burke et al. (2003b) based on a 'criss-cross' procedure. In Michiels et al. (2006), these results are extended to matrix functions of the form (2) where the perturbations take the form of (5). In particular, it is shown in Theorem 1 that the pseudospectral abscissa as defined in (4) can be expressed in the following way:

$$\Lambda_\epsilon = \left\{ \lambda \in \mathbb{C} : f(\lambda) > \frac{1}{\epsilon} \right\} \tag{8}$$

where

$$f(\lambda) = w(\lambda)\sigma_{\max}(F^{-1}(\lambda)), \quad w(\lambda) = \sum_{i=0}^{m} \frac{|e^{-\lambda\tau_i}|}{w_i}. \tag{9}$$

Using the pseudospectra definition in (8), the pseudospectral abscissa in (6) can be rewritten as

$$\alpha_\epsilon = \sup_{\lambda\in\mathbb{C}} \left\{ \Re(\lambda) : f(\lambda) = \frac{1}{\epsilon} \right\}. \tag{10}$$

The supremum in the definition (10) is well-defined since $F^{-1}(\lambda)$ is a strictly proper function and $w(\lambda)$ is uniformly bounded on any right half complex plane in (9).

In Section 2, a bisection algorithm is given for pseudospectral abscissa computation of time-delay systems based on the connection between the pseudospectra and the level sets of the function $f(\lambda)$.

This algorithm is implemented in two steps: first the approximate pseudospectral abscissa is computed by the prediction step in Section 3 and then the approximate results are corrected in Section 4.

The overall algorithm for the pseudospectral abscissa computation is outlined in Section 5. A numerical example and concluding remarks can be found in Sections 6 and 7.

Notation:

The notation in the paper is standard and given below.

$\mathbb{C}, \mathbb{R}$:	the field of the complex and real numbers,
$\mathbb{R}_0^+$:	the positive real numbers including zero,
$\Re(u)$:	real part of the complex number u,
$\Im(u)$:	imaginary part of the complex number u.
$\lvert u\rvert$:	magnitude of the complex number u.
$\bar{u}$:	conjugate of the complex number u.
A^* :	complex conjugate transpose of the matrix A.
$I_{n\times n}$:	identity matrix with dimension n.
$\sigma_{\max}(A)$:	the largest singular value of the matrix A.
$\|F(j\omega)\|_\infty$:	$\mathcal{L}_\infty$ norm of the transfer function F

2. THE BISECTION ALGORITHM FOR THE PSEUDOSPECTRAL ABSCISSA COMPUTATION

Given the function f in (9), define the function $\alpha_f(\sigma)$ as:

$$\alpha_f(\sigma) := \sup_{\omega\in\mathbb{R}} f(\sigma + j\omega) \tag{11}$$

over $\sigma \in (\alpha_0, \infty)$.

Proposition 1. The function $\alpha_f(\sigma)$ is strictly monotonically decreasing over the interval $\sigma \in (\alpha_0, \infty)$.

Proof. The proof is by contradiction. Assume that the function $\alpha_f(\sigma)$ is strictly increasing for some σ. Then there exists a level set $\frac{1}{\epsilon_1}$ such that there are at least two disjoint pseudospectra regions in the complex plane due to (8) and the strictly properness of $F^{-1}(\lambda)$ (see the blue lines in Figure 1). Since $\alpha_f(\sigma)$ is a continuous function, one of the disjoint sets in the pseudospectra disappears without merging to other pseudospectra sets for a higher level set $\frac{1}{\epsilon_3}$ (red lines in the Figure). This is a contradiction with the continuity of the individual eigenvalues and the fact

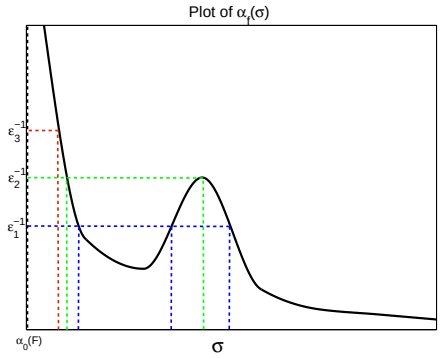

Fig. 1. $\alpha_f(\sigma)$ plot

that $F(\sigma)^{-1}$ is strictly proper (preventing eigenvalues to move off to infinity).

Similarly, it can be shown that the case $\alpha_f(\sigma_1) = \alpha_f(\sigma_2)$ with $\sigma_0 < \sigma_1 < \sigma_2$ is a contradiction with the continuity of the individual eigenvalues. □

Proposition 2. The function $\alpha_f(\sigma)$ satisfies

$$\lim_{\sigma\to\alpha_0(F)^+} \alpha_f(\sigma) = \infty,$$
$$\lim_{\sigma\to+\infty} \alpha_f(\sigma) = 0.$$

Proof. The first assertion in the Proposition follows from the fact that there are characteristic roots on the boundary $\Re(s) = \alpha_0$. Therefore, the $\mathcal{H}_\infty$ norm of $F^{-1}(\sigma + j\omega)$ diverges to infinity since the denominator becomes singular. The second assertion is the result of that F^{-1} is a strictly proper function. □

Using the definition in (10), the pseudospectral abscissa is the σ value of the function where $\alpha_f(\sigma) = \frac{1}{\epsilon}$. Since the function $\alpha_f(\sigma)$ is strictly decreasing by Proposition 1 and it attains all the values from 0 to ∞ by Proposition 2, the pseudospectral abscissa can be calculated by the following bisection algorithm.

Algorithm 1.

1) $\sigma_L = \alpha_0$, $\sigma_R = \infty$, $\Delta\sigma$ =tol,
2) while $(\sigma_R - \sigma_L) > 2 \times$ tol
   2.1) $\Delta\sigma = 2 \times \Delta\sigma$,
   2.2) if $(\sigma_R = \infty)$
        then $\sigma_M = \sigma_L + \Delta\sigma$,
        else $\sigma_M = \frac{\sigma_L + \sigma_R}{2}$.
   2.3) determine if $\alpha_f(\sigma_M) > \frac{1}{\epsilon}$
        then $\sigma_L = \sigma_M$,
        else $\sigma_R = \sigma_M$.

{result: the approximate pseudospectral abscissa , $\tilde{\sigma} = \sigma_L$}

The main computation in the bisection algorithm is to check whether the inequality in step 2.3 is satisfied. By algebraic computation, the function $\alpha_f(\sigma)$ is equivalent to

$$\begin{aligned}\alpha_f(\sigma) &= \sup_{\omega\in\mathbb{R}} f(\sigma + j\omega),\\ &= \sup_{\omega\in\mathbb{R}} \left\{ w(\sigma + j\omega)\sigma_{\max}(F^{-1}(\sigma + j\omega)) \right\},\\ &= w(\sigma) \sup_{\omega\in\mathbb{R}} \left\{ \sigma_{\max}(F^{-1}(\sigma + j\omega)) \right\},\\ &= w(\sigma)\|F_\sigma^{-1}\|_\infty \end{aligned} \tag{12}$$

where

$$F_\sigma^{-1}(j\omega) = \left(j\omega I_{n\times n} - \left(\sum_{i=0}^{m} A_{\sigma,i} e^{-j\omega\tau_i} \right) \right)^{-1} \tag{13}$$

and

$$\begin{aligned} A_{\sigma,0} &= A_0 - \sigma I_{n\times n}, \\ A_{\sigma,i} &= A_i e^{-\tau_i \sigma}, \text{ for } i = 1, \ldots, m. \end{aligned} \tag{14}$$

The inequality $\alpha_f(\sigma) > \frac{1}{\epsilon}$ in step 2.3 is satisfied if and only if $F_\sigma^{-1}(j\omega)$ has a singular value equal to $\frac{1}{\epsilon w(\sigma)}$ for some value of ω. This condition can be reduced into the verification of the imaginary axis eigenvalues of an infinite dimensional operator as shown in the following Theorem.

Theorem 3. The inequality

$$\alpha_f(\sigma) > \frac{1}{\epsilon} \tag{15}$$

is satisfied if and only if the linear infinite dimensional operator $\mathcal{L}_\sigma$ has eigenvalues on the imaginary axis where $\mathcal{L}_\sigma$ is defined on $X := \mathcal{C}([-\tau_{\max},\ \tau_{\max}], \mathbb{C}^{2n})$ by

$$\begin{aligned} \mathcal{D}(\mathcal{L}_\sigma) = \{ &\phi \in X :\ \phi' \in X, \\ &\phi'(0) = M_0\phi(0) + \sum_{i=1}^{m} (M_i\phi(-\tau_i) + M_{-i}\phi(\tau_i))\}, \end{aligned} \tag{16}$$

$$\mathcal{L}_\sigma \phi = \phi',\ \phi \in \mathcal{D}(\mathcal{L}_\sigma) \tag{17}$$

with

$$\begin{aligned} M_0 &= \begin{bmatrix} A_{\sigma,0} & -(\epsilon w(\sigma))^2 I_{n\times n} \\ -I_{n\times n} & -A_{\sigma,0}^* \end{bmatrix}, \\ M_i &= \begin{bmatrix} A_{\sigma,i} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{-i} = \begin{bmatrix} 0 & 0 \\ 0 & -A_{\sigma,i}^* \end{bmatrix}, \quad 1 \le i \le N. \end{aligned}$$

Proof. A similar proof is given in Gumussoy and Michiels (2009). This Theorem generalizes Proposition 28 of Genin et al. (2002). □

Using Theorem 3, we can refine the *conceptual* algorithm for the pseudospectral abscissa computation,

Algorithm 2.

1) $\sigma_L = \alpha_0$, $\sigma_R = \infty$, $\Delta\sigma$ =tol,
2) while $(\sigma_R - \sigma_L) > 2 \times$ tol
   2.1) $\Delta\sigma = 2 \times \Delta\sigma$,
   2.2) if $(\sigma_R = \infty)$
        then $\sigma_M = \sigma_L + \Delta\sigma$,
        else $\sigma_M = \frac{\sigma_L + \sigma_R}{2}$.
   2.3) if $\mathcal{L}_{\sigma_M}$ has imaginary axis eigenvalues
        then $\sigma_L = \sigma_M$,
        else $\sigma_R = \sigma_M$.

{result: the approximate pseudospectral abscissa , $\tilde{\sigma} = \sigma_L$}

Note that step 2.3 in the Algorithm 3 requires solving a linear infinite dimensional eigenvalue problem for $\mathcal{L}_\sigma$ which needs to be discretized in a practical implementation. We will do this using a *spectral method* (see, e.g. Trefethen (2000); Breda et al. (2005, 2006)) and calculate the approximate solution by solving the standard linear eigenvalue problem. This approach is described in the next section.

3. PREDICTING THE PSEUDOSPECTRAL ABSCISSA

Given a positive integer N, we consider a mesh Ω_N of $2N+1$ distinct points in the interval $[-\tau_{\max},\ \tau_{\max}]$:

$$\Omega_N = \{\theta_{N,i},\ i = -N, \ldots, N\}, \tag{18}$$

where

$$\begin{aligned} -\tau_{\max} \le \theta_{N,-N} < \ldots < \theta_{N,-1} < \theta_{N,0} = 0 \\ < \theta_{N,1} < \cdots < \theta_{N,N} \le \tau_{\max}. \end{aligned}$$

This allows to replace the continuous space X with the space X_N of discrete functions defined over the mesh Ω_N, i.e. any function $\phi \in X$ is discretized into a block vector $x = [x_{-N}^T \cdots\ x_N^T]^T \in X_N$ with components

$$x_i = \phi(\theta_{N,i}) \in \mathbb{C}^{2n}, \quad i = -N, \ldots, N.$$

Let $\mathcal{P}_N x,\ x \in X_N$ be the unique $\mathbb{C}^{2n}$ valued interpolating polynomial of degree $\le 2N$ satisfying

$$\mathcal{P}_N x(\theta_{N,i}) = x_i, \quad i = -N, \ldots, N.$$

In this way, the operator $\mathcal{L}_\sigma$ over X can be approximated with the matrix $\mathcal{L}_\sigma^N :\ X_N \to X_N$, defined as

$$\begin{aligned} \left(\mathcal{L}_\sigma^N\ x\right)_i &= \left(\mathcal{P}_N x\right)'(\theta_{N,i}), \quad i = -N, \ldots, -1, \\ \left(\mathcal{L}_\sigma^N\ x\right)_0 &= M_0 \mathcal{P}_N x(0) + \sum_{i=1}^{m} (M_i \mathcal{P}_N x(-\tau_i) + M_{-i} \mathcal{P}_N x(\tau_i)) \\ \left(\mathcal{L}_\sigma^N\ x\right)_i &= \left(\mathcal{P}_N x\right)'(\theta_{N,i}), \quad i = 1, \ldots, N. \end{aligned} \tag{19}$$

Using the Lagrange representation of $\mathcal{P}_N x$,

$$\mathcal{P}_N x = \sum_{k=-N}^{N} l_{N,k}\ x_k, \quad ,$$

where the Lagrange polynomials $l_{N,k}$ are real valued polynomials of degree $2N$ satisfying

$$l_{N,k}(\theta_{N,i}) = \begin{cases} 1 & i = k, \\ 0 & i \ne k, \end{cases}$$

we obtain the explicit form

$$\mathcal{L}_\sigma^N = \begin{bmatrix} d_{-N,-N} & \ldots & d_{-N,N} \\ \vdots & & \vdots \\ d_{-1,-N} & \ldots & d_{-1,N} \\ a_{-N} & \ldots & a_N \\ d_{1,-N} & \ldots & d_{1,N} \\ \vdots & & \vdots \\ d_{N,-N} & \ldots & d_{N,N} \end{bmatrix} \in \mathbb{R}^{(2N+1)(2n)\times(2N+1)2n},$$

where

$$\begin{aligned} d_{i,k} &= l'_{N,k}(\theta_{N,i}) I, \quad i,k \in \{-N, \ldots, N\},\ i \ne 0 \\ a_0 &= M_0\ x_0 + \sum_{i=1}^{m} (M_i l_{N,0}(-\tau_i) + M_{-i} l_{N,0}(\tau_i)) \\ a_k &= \sum_{i=1}^{m} (M_i l_{N,k}(-\tau_i) + M_{-i} l_{N,k}(\tau_i)) \\ &\qquad\qquad k \in \{-N, \ldots, N\},\ k \ne 0. \end{aligned}$$

Note that all the problem specific information and the parameter σ are concentrated in the middle row of $\mathcal{L}_\sigma^N$, i.e. the elements $(a_{-N}, \ldots, a_N)$, while all other elements of $\mathcal{L}_\sigma^N$ can be computed beforehand.

Since step 2.3 of Algorithm 2 is based on checking the presence of eigenvalues of $\mathcal{L}_\sigma$ on the imaginary axis and

thus strongly rely on the symmetry of the eigenvalues with respect to the imaginary axis, it is important that this property is *preserved* in the discretization. The following Proposition gives the condition on the mesh such that this symmetry holds.

Proposition 4. If the mesh Ω_N satisfies

$$\theta_{N,-i} = -\theta_{N,i},\ i = 1, \ldots, N, \tag{20}$$

then the following result hold: for all $\lambda \in \mathbb{C}$, we have

$$\det\left(\lambda I - \mathcal{L}_\sigma^N\right) = 0 \Leftrightarrow \det\left(-\bar{\lambda} - \mathcal{L}_\sigma^N\right) = 0. \tag{21}$$

Proof. Consider the differentiation matrix with elements

$$\Delta_{k,l} = l'_{N,k-N-1}(\theta_{N,l-N-1}),\quad k, l \in \{1, \ldots, 2N+1\}.$$

and let U be such that $U^{-1}\Delta U = \Delta^T$. Define the matrix $S \in \mathbb{R}^{2N+1\times 2N+1}$ with terms equal to 1 on the main skew diagonal and 0 elsewhere, the symmetry property of the mesh (20) assures that

$$\left(U^{-1}S\right) \otimes \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \mathcal{L}_\sigma^N \left(US\right) \otimes \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} = -\left(\mathcal{L}_\sigma^N\right)^*,$$

that is, the matrices $\mathcal{L}_\sigma^N$ and $-\left(\mathcal{L}_\sigma^N\right)^T$ are similar. The proposition directly follows. □

Based on the discretization of $\mathcal{L}_\sigma$ into $\mathcal{L}_\sigma^N$, we propose the following algorithm to approximate (predict) the pseudospectral abscissa. It corresponds to Algorithm 2 where step 2.3 is replaced with the matrix $\mathcal{L}_\sigma^N$ for a fixed N.

Algorithm 3.

1) $\sigma_L = \alpha_0$, $\sigma_R = \infty$, $\Delta\sigma$ =tol,
2) while $(\sigma_R - \sigma_L) > 2 \times$ tol
   2.1) $\Delta\sigma = 2 \times \Delta\sigma$,
   2.2) if $(\sigma_R = \infty)$
        then $\sigma_M = \sigma_L + \Delta\sigma$,
        else $\sigma_M = \frac{\sigma_L + \sigma_R}{2}$.
   2.3) if $\mathcal{L}_{\sigma_M}^N$ has imaginary axis eigenvalues
        then $\sigma_L = \sigma_M$,
        else $\sigma_R = \sigma_M$.

{result: the approximate pseudospectral abscissa , $\tilde{\sigma} = \sigma_L$}

In what follows we clarify the effect of using the discretized operator in the algorithm. The next Theorem establishes the link between the imaginary axis eigenvalues of $\mathcal{L}_\sigma^N$ and the corresponding inequality check condition similar to the connection between $\mathcal{L}_\sigma$ and the inequality (15) in Theorem 3.

Theorem 5. Assume that the mesh Ω_N is symmetric around the zero as given in (20). Let p_N be the polynomial of the degree $2N$ satisfying the conditions,

$$\begin{aligned} &p_N(0;\ \lambda) = 1, \\ &p'_N(\theta_i;\lambda) = \lambda p_N(\theta_i;\lambda),\ i = -N, \ldots, -1, 1, \ldots, N. \end{aligned} \tag{22}$$

The matrix $\mathcal{L}_\sigma^N$ has an imaginary axis eigenvalue $\lambda = j\omega$ if and only if the inequality

$$\alpha_f^N(\sigma) > \frac{1}{\epsilon} \tag{23}$$

holds where

$$\alpha_f^N(\sigma) := \sup_{\omega\in\mathbb{R}} f_N(\sigma + j\omega) \text{ and}$$

$$f_N(\sigma + j\omega) = w(\sigma)\left(j\omega I - A_{\sigma,0} - \sum_{i=1}^m A_{\sigma,i} p_N(-\tau_i;\ j\omega)\right)^{-1}$$

Therefore, the effect of using $\mathcal{L}_\sigma^N$ instead of $\mathcal{L}_\sigma$ corresponds to computing the *approximate* pseudospectral abscissa

$$\alpha_\epsilon^N = \sup_{\lambda\in\mathbb{C}} \left\{ \Re(\lambda) : f_N(\lambda) = \frac{1}{\epsilon} \right\}.$$

The accuracy of the approximation depends on the discretization parameter N. Therefore, the accuracy can be chosen arbitrarily close to α_ϵ by increasing N. Note that at each iteration of step 2.3 in Algorithm 3, an eigenvalue problem of size $(2n)(2N+1)$ needs to be solved which may be computationally very demanding for large N.

However, because the eigenvalues of $\mathcal{L}_\sigma^N$ exhibit the spectral convergence to the corresponding eigenvalues of $\mathcal{L}_\sigma$ (following the lines of Breda et al. (2005)) and because the approximation error can be corrected in a very cheap way as we shall see in the next section, it is sufficient to have a small value of N for most practical applications.

4. CORRECTING THE PSEUDOSPECTRAL ABSCISSA

The Bisection Algorithm 3 finds the complex points $\tilde{\lambda}_i = \tilde{\sigma} + j\tilde{\omega}_i$ for $i = 1, \ldots, \tilde{n}$ achieving the approximate pseudospectral abscissa $\tilde{\sigma}$ which is close to the α_ϵ given tolerance and discretization points N. These approximate results are corrected by using the property that the eigenvalues of the pseudospectral abscissa appear as solutions of a finite dimensional nonlinear eigenvalue problem. The following theorem establishes the link between this nonlinear eigenvalue problem and the linear eigenvalue problem of $\mathcal{L}_\sigma$.

Theorem 6. λ is an eigenvalue of linear operator $\mathcal{L}_\sigma$ if and only if

$$\det H_\sigma(\lambda) = 0, \tag{24}$$

where

$$H_\sigma(\lambda) := \lambda I - M_0 - \sum_{i=1}^m \left(M_i e^{-\lambda\tau_i} + M_{-i} e^{\lambda\tau_i}\right) \tag{25}$$

and the matrices M_0, M_i, M_{-i} are defined in Theorem 3.

The solutions of (24) can be found by solving

$$H_\sigma(\lambda)\ v = 0,\quad \lambda \in \mathbb{C},\ v \in \mathbb{C}^{2n},\ v \neq 0, \tag{26}$$

which in general has an infinite number of solution.

The correction method is based on the property that if $\omega(\sigma)\|F_\sigma^{-1}(j\omega)\|_\infty = \frac{1}{\epsilon}$, then the operator $\mathcal{L}_\sigma$, or equivalently, (26) has a multiple non-semisimple eigenvalue as shown in Figure 2:

If $\lambda_\epsilon = \alpha_\epsilon + j\omega_\epsilon$ are such that

$$w(\sigma)\|F_\sigma^{-1}(j\omega)\|_\infty = \frac{1}{\epsilon} = w(\alpha_\epsilon)\sigma_{\max}\left(F_{\alpha_\epsilon}^{-1}(j\omega_\epsilon)\right), \tag{27}$$

then setting

$$h_\sigma(\lambda) = \det H_\sigma(\lambda),$$

the pair $(\omega, \alpha) = (\omega_\epsilon, \alpha_\epsilon)$ satisfies

$$h_\sigma(j\omega) = 0,\quad h'_\sigma(j\omega) = 0. \tag{28}$$

These complex-valued equations seem over-determined but this is not the case due to the spectral properties of $H_\sigma(\lambda)$. Using the symmetry of the eigenvalues of the nonlinear eigenvalue problem (26) with respect to imaginary axis, we can write the following:

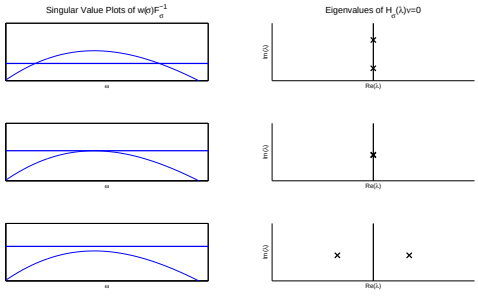

Fig. 2. (left) Intersections of the singular value plot of $w(\sigma)F_\sigma^{-1}$ with the horizontal line $\frac{1}{\epsilon}$ for the cases where (top) $w(\sigma)\|F_\sigma^{-1}(j\omega)\|_\infty > \frac{1}{\epsilon}$, (middle) $w(\sigma)\|F_\sigma^{-1}(j\omega)\|_\infty = \frac{1}{\epsilon}$ and (bottom) $w(\sigma)\|F_\sigma^{-1}(j\omega)\|_\infty < \frac{1}{\epsilon}$. (right) Corresponding eigenvalues of the problem (26).

Corollary 7. For $\omega \geq 0$, we have

$$\Im\ h_\sigma(j\omega) = 0 \tag{29}$$

and

$$\Re\ h'_\sigma(j\omega) = 0. \tag{30}$$

Proof. From the symmetry property of the eigenvalues with respect to the imaginary axis,

$$h_\sigma(\lambda) = h_\sigma(-\lambda), \quad h'_\sigma(\lambda) = -h'_\sigma(-\lambda).$$

Substituting $\lambda = j\omega$ yields

$$\begin{aligned} h_\sigma(j\omega) &= h_\sigma(-j\omega) = (h_\sigma(j\omega))^*, \\ h'_\sigma(j\omega) &= -h'_\sigma(-j\omega) = -(h'_\sigma(j\omega))^*, \end{aligned}$$

and the assertions follow. □

Using Corollary 7 we can simplify the conditions (28) to:

$$\begin{cases} \Re\ h_\sigma(j\omega) = 0 \\ \Im\ h'_\sigma(j\omega) = 0 \end{cases}. \tag{31}$$

Hence, the pair $(\omega_\epsilon, \alpha_\epsilon)$ satisfying (27) can be directly computed from the two equations (31), e.g. using Newton's method, provided that good starting values are available.

The drawback of working directly with (31) is that an explicit expression for the determinant of H_σ is required. To avoid this, let $u, v \in \mathbb{C}^n$ be such that

$$H_\sigma(j\omega)\begin{bmatrix} u \\ v \end{bmatrix} = 0, \quad n(u,v) = 0, \tag{32}$$

where $n(u,v) = 0$ is a normalizing condition. Given the structure of H_σ it can be verified that a corresponding left eigenvector is given by $[-v^*\ u^*]$. According to Hryniv and Lancaster (1999), we get

$$h'_\sigma(j\omega) = 0 \Leftrightarrow [-v^*\ u^*]\ H'_\sigma(j\omega)\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

A simple computation yields:

$$[-v^*\ u^*]\ H'_\sigma(j\omega)\begin{bmatrix} u \\ v \end{bmatrix} = 2\Im\left\{v^*\left(I + \sum_{i=1}^{p} A_{\sigma,i}\tau_i e^{-j\omega\tau_i}\right)u\right\}, \tag{33}$$

which is always real. This is a consequence of the property (30).

Taking into account the above results, we end up with $4n+3$ real equations

$$\begin{cases} H_\sigma(j\omega,\ \sigma)\begin{bmatrix} u, \\ v \end{bmatrix} = 0, \quad n(u,v) = 0 \\ \Im\left\{v^*\left(I + \sum_{i=1}^{p} A_{\sigma,i}\tau_i e^{-j\omega\tau_i}\right)u\right\} = 0 \end{cases} \tag{34}$$

in the $4n+2$ unknowns $\Re(v), \Im(v), \Re(u), \Im(u), \omega$ and σ. These equations are still overdetermined because the property (29) is not explicitly exploited in the formulation, unlike the property (30). However, it makes the equations (34) exactly solvable, and the (ω, σ) components have a one-to-one-correspondence with the solutions of (31).

In conclusion, as a result of the bisection algorithm in the prediction step, the approximate pseudospectral abscissa $\tilde{\sigma}$ and the corresponding critical frequencies $\tilde{\omega}_i$ for $i = 1, \ldots, \tilde{n}$ are calculated. Note that these computations are based on the approximation of $\mathcal{L}_\sigma$ into a matrix $\mathcal{L}_\sigma^N$. Using these approximate results as estimates of $(\omega_\epsilon, \alpha_\epsilon)$ (27), we can compute the approximate eigenvectors u and v. These approximate values improved in the correction step by solving (34). At the end of the correction step, the pseudospectral abscissa $\sigma = \alpha_\epsilon$ and the achieved frequency $\omega = \omega_\epsilon$ are obtained within predefined tolerance.

5. ALGORITHM

The overall algorithm for computing the pseudospectral abscissa consists of two steps: the prediction step and the correction step. The first step requires a repeated computation of the eigenvalues of a $(2N+1)2n \times (2N+1)2n$ matrix $\mathcal{L}_\sigma^N$. The second step solves (34) with $4n+3$ equations and $4n+2$ unknowns using Gauss-Newton algorithm. Our method chooses N sufficiently large such that the results of the prediction step are good starting values for the correction step. Note that by increasing N and using only the prediction step, the approximate pseudospectral abscissa can be computed arbitrarily close to α_ϵ. However, this approach has more numerical cost than the combined approach when N is large.

Algorithm 4.
Input: system data, tolerance tol for prediction step, discretization points N
Output: pseudospectral abscissa α_ϵ

Prediction Step:

1) Calculate the spectral abscissa α_0 of F (2),
2) $\sigma_L = \alpha_0$, $\sigma_R = \infty$, $\Delta\sigma$ =tol,
3) while $(\sigma_R - \sigma_L) > 2 \times$ tol
   3.1) $\Delta\sigma = 2 \times \Delta\sigma$,
   3.1) if $(\sigma_R = \infty)$
   then $\sigma_M = \sigma_L + \Delta\sigma$,
   else $\sigma_M = \frac{\sigma_L + \sigma_R}{2}$.
   3.2) if $\mathcal{L}_{\sigma_M}^N$ has imaginary axis eigenvalues
   then $\sigma_L = \sigma_M$,
   else $\sigma_R = \sigma_M$.

{result: the approximate pseudospectral abscissa , $\tilde{\sigma} = \sigma_L$ and the corresponding frequencies $j\tilde{\omega}_i$ $i = 1, \ldots, \tilde{n}$ of $\mathcal{L}_\sigma^N$}

Correction Step:

(1) calculate the approximate null vectors $\{x_1, \ldots, x_{\tilde{n}}\}$ of $H_{\tilde{\sigma}}(j\tilde{\omega}_i)$ $i = 1, \ldots, \tilde{n}$,

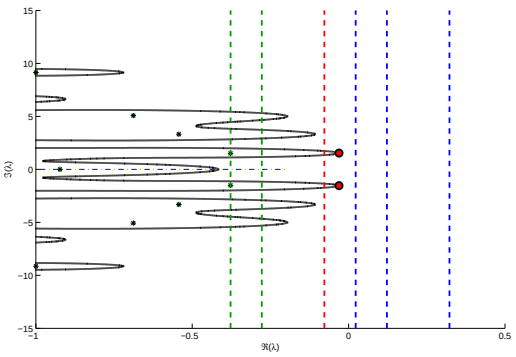

Fig. 3. The pseudospectra and the pseudospectral abscissa

(2) for all $i \in \{1, \ldots, \tilde{n}\}$, solve (34) with starting values

$$\begin{bmatrix} u \\ v \end{bmatrix} = x_i, \ \omega = \tilde{\omega}_i, \quad \sigma = \tilde{\sigma}$$

denote the solution with $(u_{\epsilon,i}, v_{\epsilon,i}, \omega_{\epsilon,i}, \sigma_{\epsilon,i})$.

(3) set $\alpha_\epsilon := \max_{1 \le i \le \tilde{n}} \sigma_{\epsilon,i}$.

In our implementation, mesh points are chosen as Chebyshev extremal points since the corresponding interpolation polynomial has less oscillation towards the end of the interval compared to another distribution of mesh points, Breda et al. (2006).

Note that the spectral abscissa calculation in the prediction step requires the calculation of the right-most eigenvalue of time-delay systems. This computation is done by DDE-BIFTOOL, Engelborghs et al. (2002). The overall algorithm is fully automated and implemented as a MATLAB function.

6. EXAMPLE

We tested the numerical method on several benchmark problems. We generated the following difficult example to benchmark our method. We consider a time-delay system F in (2) with the dimensions $m = 7$, $n = 10$, $n_u = 2$, $n_y = 4$ with delays $\tau_1 = 0.1$, $\tau_2 = 0.2$, $\tau_3 = 0.3$, $\tau_4 = 0.4$, $\tau_5 = 0.5$, $\tau_6 = 0.6$, $\tau_7 = 0.8$. The weights w_i are set to 1 and $\epsilon = 0.1$. The pseudospectra is shown with black lines and black stars indicate the characteristic roots of (1) in Figure 3.

The tolerance in the bisection algorithm is set to 0.05 and the discretization parameter is chosen as $N = 6$. Each iteration of the while loop in the prediction step computes σ_M and updates σ_L or σ_R shown as the vertical green and blue lines respectively. The approximate pseudospectral abscissa as a result of the prediction step is $\tilde{\sigma} = -0.0774$ and the corresponding critical frequencies are $\tilde{\omega}_1 = 1.3493$, $\tilde{\omega}_2 = 1.7318$.

These approximate values are improved in the correction step and the computed pseudospectral abscissa is $\alpha_\epsilon = -0.0307$ at $\omega_\epsilon = 1.5383$ shown as red dots in Figure 3.

7. CONCLUDING REMARKS

An accurate method to compute the pseudospectral abscissa of retarded time-delay systems with arbitrary number of delays is given. The method is based on two steps: The prediction step calculates the approximate pseudospectral abscissa using the connection between pseudospectra and the level set of a function. The correction step computes the pseudospectral abscissa by solving equations based on the nonlinear eigenvalue problem. The method is successfully applied to the moderate size example and its effectiveness is shown.

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