

Remarks on Strong Stabilization and Stable \mathcal{H}^∞ Controller Design*

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Abstract—A state space based design method is given to find strongly stabilizing controllers for multi-input-multi-output plants (MIMO). A sufficient condition is derived for the existence of suboptimal stable \mathcal{H}^∞ controller in terms of linear matrix inequalities (LMI) and the controller order is twice that of the plant. A new parameterization of strongly stabilizing controllers is determined using linear fractional transformations (LFT).

I. INTRODUCTION

Strong stabilization problem is known as the design of a stable feedback controller which stabilizes the given plant. For practical reasons, a stable controller is desired [1], [2]. In this paper, we derive a simple and effective design method to find stable \mathcal{H}^∞ controllers for MIMO systems.

A stable controller can be designed if and only if the plant satisfies the parity interlacing property (PIP) [3] i.e., the plant has even number of poles between any pair of its zeros on the extended positive real axis. There are several design procedure for strongly stabilizing controllers, [4–16].

The result in this paper is the generalization of the work in [11] using LMIs. The procedure is quite simple, efficient and easy to solve by using the LMI Toolbox of MATLAB [17]. In the next section, it is shown that if a certain LMI has a feasible solution, then it is possible to obtain a stable \mathcal{H}^∞ controller whose order is twice the order of the plant. Moreover, a parameterization of strongly stabilizing controllers can be given in terms of LFT.

The paper is organized as follows. The main results are given in Section 2. Stable \mathcal{H}^∞ controller design procedure is proposed in Section 3. Numerical examples and comparison with other methods can be found in Section 4 and concluding remarks are made in the last section.

Notation

The notation is fairly standard. A state space realization of a transfer function, $G(s) = C(sI - A)^{-1}B + D$, is shown by $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ and the linear fractional transformation of G by K is denoted by $\mathcal{F}_l(G, K)$ which is equivalent to $G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$ where G is partitioned as $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$. As a shorthand notation for LMI

expressions, we will define $\Gamma(A, B) := B^T A^T + AB$ where A, B are matrices with compatible dimensions.

II. STRONG STABILIZATION OF MIMO SYSTEMS

Consider the standard feedback system with generalized plant, G , which has state space realization,

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_{11} & D_{12} \\ C & D_{21} & 0 \end{array} \right] \quad (\text{II.1})$$

where $A \in \mathcal{R}^{n \times n}$, $D_{12} \in \mathcal{R}^{p_1 \times m_2}$, $D_{21} \in \mathcal{R}^{p_2 \times m_1}$ and other matrices are compatible with each other. We suppose the plant satisfies the standard assumptions,

- A.1 (A, B) is stabilizable and (C, A) is detectable,
- A.2 $\begin{bmatrix} A - \lambda I & B \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\text{Re}\{\lambda\} \geq 0$,
- A.3 $\begin{bmatrix} A - \lambda I & B_1 \\ C & D_{21} \end{bmatrix}$ has full row rank for all $\text{Re}\{\lambda\} \geq 0$,
- A.4 A has no eigenvalues on the imaginary axis.

Let the controller has state space realization, $K_G(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & 0 \end{array} \right]$ where $A_K \in \mathcal{R}^{n \times n}$, $B_K \in \mathcal{R}^{n \times p_2}$ and $C_K \in \mathcal{R}^{m_2 \times n}$. Define the matrix $X \in \mathcal{R}^{n \times n}$, $X = X^T > 0$ as the stabilizing solution of

$$A^T X + X A - X B B^T X = 0 \quad (\text{II.2})$$

(i.e., $A - B B^T X$ is stable) and the “A-matrix” of the closed loop system as $A_{CL} = \begin{bmatrix} A & B C_K \\ B_K C & A_K \end{bmatrix}$. Note that since (A, B) is stabilizable, X is unique and $A_X := (A - B B^T X)$ is stable. Also, the closed loop stability is equivalent to whether A_{CL} is stable or not.

Lemma 2.1: Assume that the plant (II.1) satisfies the assumptions A.1 – A.4. There exists a stable stabilizing controller, $K_G \in \mathcal{RH}^\infty$ if there exists $X_K \in \mathcal{R}^{n \times n}$, $X_K = X_K^T > 0$ and $Z \in \mathcal{R}^{n \times p_2}$ for some $\gamma_K > 0$ satisfying the LMIs,

$$\Gamma(X_K, A) + \Gamma(Z, C) < 0, \quad (\text{II.3})$$

$$\begin{bmatrix} \Gamma(X_K, A_X) + \Gamma(Z, C) & -Z & -X B \\ -Z^T & -\gamma_K I & 0 \\ -B^T X & 0 & -\gamma_K I \end{bmatrix} < 0, \quad (\text{II.4})$$

where X is the stabilizing solution of (II.2) and A_X is as defined previously. Moreover, under the above condition, a stable controller can be given as $K_G(s) =$

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$\left[\begin{array}{c|c} A_X + X_K^{-1}ZC & -X_K^{-1}Z \\ \hline -B^T X & 0 \end{array} \right]$ and this controller satisfies $\|K_G\|_\infty < \gamma_K$.

Proof: By using similarity transformation, one can show that A_{CL} is stable if and only if A_X and $A_Z := A + X_K^{-1}ZC$ is stable. Since X is a stabilizing solution, A_X is stable. If we rewrite the LMI (II.3) as

$$(A + X_K^{-1}ZC)^T X_K + X_K(A + X_K^{-1}ZC) < 0,$$

it can be seen that A_Z is stable since $X_K > 0$. The second LMI (II.4) comes from KYP lemma and guarantees that $\|K_G\|_\infty < \gamma_K$. ■

Remark 1 If the design only requires the stability of closed loop system, it is enough to satisfy the LMI (II.3), (1,1) block of (II.4), i.e.,

$$A_X^T X_K + X_K A_X + C^T Z^T + ZC < 0 \quad (\text{II.5})$$

and the controller has same structure as above.

Remark 2 The Lemma (2.1) is generalization of Theorem 2.1 in [11]. If the algebraic riccati equation (ARE) (7) in [11] has a stabilizing solution, $Y = Y^T \geq 0$, then there exists a stable controller in the form, $\left[\begin{array}{c|c} A_X - \gamma_K^2 Y C^T C & \gamma_K^2 Y C^T \\ \hline -B^T X & 0 \end{array} \right]$. This structure is the special case of the LMIs (II.3) and (II.4) when $X_K = (\gamma_K Y)^{-1}$ and $Z = -\gamma_K C^T$. Note that our formulation does not assume special structure on Z . Also in [11], the stability of A_Z is guaranteed by the same riccati equation, we satisfy the stability condition of A_Z with another LMI (II.3) which is less restrictive. Therefore, the Lemma (2.1) is less conservative as will be demonstrated in examples.

Corollary 2.1: Assume that the sufficient condition (II.3) and (II.5) holds. Then all controllers in the set

$$K_{G,ss} := \{K = \mathcal{F}_l(K_{G,ss}^0, Q) : Q \in \mathcal{RH}^\infty, \|Q\|_\infty < \gamma_Q\}$$

are strongly stabilizing where

$$K_{G,ss}^0(s) = \left[\begin{array}{c|cc} A_X + X_K^{-1}ZC & -X_K^{-1}Z & B \\ \hline -B^T X & 0 & I \\ -C & I & 0 \end{array} \right]. \quad (\text{II.6})$$

and $\gamma_Q = (\|C(sI - (A_X + X_K^{-1}ZC))^{-1}B\|_\infty)^{-1}$.

Proof: The result is direct consequence of parameterization of all stabilizing controllers [19]. ■

III. STABLE \mathcal{H}^∞ CONTROLLER DESIGN FOR MIMO SYSTEMS

The standard \mathcal{H}^∞ problem is to find a stabilizing controller K such that $\|\mathcal{F}_l(P, K)\|_\infty \leq \gamma$ where $\gamma > 0$ is the closed loop performance level and P is the generalized plant. It is well known that if two AREs have unique positive semidefinite solutions and the spectral radius condition is satisfied, then standard \mathcal{H}^∞ problem is solvable.

All suboptimal \mathcal{H}^∞ controllers can be parameterized as $K = \mathcal{F}_l(M_\infty, Q)$ where the central controller is in the form

$$M_\infty(s) = \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & 0 \end{array} \right]$$

and Q is free parameter satisfying $Q \in \mathcal{RH}^\infty$ and $\|Q\|_\infty \leq \gamma$. For derivation and calculation of M_∞ , see [18], [19].

If we consider M_∞ as plant and $\gamma = \gamma_K$, by using Lemma (2.1), we can find a strictly proper stable K_{M_∞} stabilizing M_∞ and resulting stable \mathcal{H}^∞ controller, $C_\gamma = \mathcal{F}_l(M_\infty, K_{M_\infty})$ where $\|K_{M_\infty}\|_\infty < \gamma_K$. If sufficient conditions (II.3) and (II.4) are satisfied, then K_{M_∞} can be written as,

$$K_{M_\infty}(s) = \left[\begin{array}{c|c} A_c - B_{c2}B_{c2}^T X_c + X_K^{-1}Z_c C_{c2} & -X_K^{-1}Z_c \\ \hline -B_{c2}^T X_c & 0 \end{array} \right]$$

and by similarity transformation, we can obtain the state space realization of C_γ as,

$$C_\gamma(s) = \left[\begin{array}{c|c} A_{C_\gamma} & B_{C_\gamma} \\ \hline C_{C_\gamma} & D_{c11} \end{array} \right]$$

where X_c is the stabilizing solution of

$$A_c^T X_c + X_c A_c - X_c B_{c2} B_{c2}^T X_c = 0$$

as in (II.2) and X_{K_c} , Z_c are the solution of (II.3), (II.4) respectively and the matrices,

$$\begin{aligned} A_{C_\gamma} &= \begin{bmatrix} A_c - B_{c2}B_{c2}^T X_c & -B_{c2}B_{c2}^T X_c \\ 0 & A_c + X_K^{-1}Z_c C_{c2} \end{bmatrix} \\ B_{C_\gamma} &= \begin{bmatrix} B_{c1} \\ -B_{c1} - X_K^{-1}Z_c D_{c21} \end{bmatrix} \\ C_{C_\gamma} &= \begin{bmatrix} C_{c1} - D_{c12}B_{c2}^T X_c & -D_{c12}B_{c2}^T X_c \end{bmatrix} \end{aligned}$$

Note that C_γ is stable stabilizing controller such that $\|\mathcal{F}_l(P, C_\gamma)\|_\infty < \gamma$.

IV. NUMERICAL EXAMPLES AND COMPARISONS

A. Strong stabilization

The numerical example is chosen from [11]. In order to see the performance of our method, we calculated the minimum γ_K satisfying the sufficient conditions in Lemma (2.1) for the following plants:

$$\begin{aligned} G_1(s) &= \left[\begin{array}{c} \frac{(s+5)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-\alpha)(s-20)} \\ \frac{(s+1)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-\alpha)(s-20)} \\ \frac{(s+1)(s-2-j\alpha)(s-2+j\alpha)}{(s+2+j)(s+2-j)(s-1)(s-5)} \\ \frac{(s+5)(s-2-j\alpha)(s-2+j\alpha)}{(s+2+j)(s+2-j)(s-1)(s-5)} \end{array} \right] \\ G_2(s) &= \left[\begin{array}{c} \frac{(s+5)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-\alpha)(s-20)} \\ \frac{(s+1)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-\alpha)(s-20)} \\ \frac{(s+1)(s-2-j\alpha)(s-2+j\alpha)}{(s+2+j)(s+2-j)(s-1)(s-5)} \\ \frac{(s+5)(s-2-j\alpha)(s-2+j\alpha)}{(s+2+j)(s+2-j)(s-1)(s-5)} \end{array} \right] \end{aligned}$$

For various α values, the minimum γ_K is found. Figure 1 and 2 illustrates the conservatism of [11] mentioned in Remark 2 (where ρ_{min} is the minimum value of the free parameter γ_K corresponding to the method of [11]).

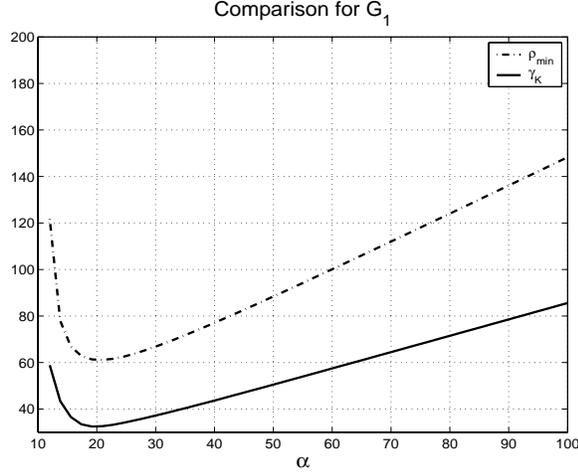


Fig. 1. Comparison for plant G_1

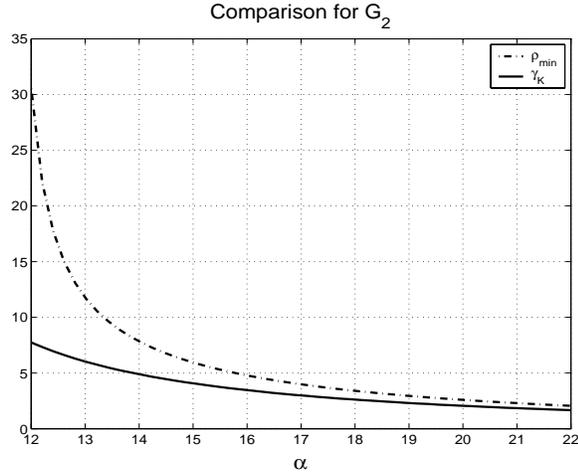


Fig. 2. Comparison for plant G_2

B. Stable \mathcal{H}^∞ controllers

We applied our method to stable \mathcal{H}^∞ controller design. As a common benchmark example, the following system is taken from [15]:

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] \quad (\text{IV.7})$$

where

$$A = \begin{bmatrix} -2 & 1.7321 \\ 1.7321 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} B_1 & B_2 & \\ \hline 0.1 & -0.1 & 1 \\ -0.5 & 0.5 & 0 \end{array} \right]$$

$$\left[\begin{array}{c} C_1 \\ C_2 \end{array} \right] = \left[\begin{array}{cc|c} 0.2 & -1 & \\ \hline 0 & 0 & \\ 10 & 11.5470 & \end{array} \right]$$

$$\left[\begin{array}{cc|c} D_{11} & D_{12} & \\ \hline D_{21} & 0 & \end{array} \right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0.7071 & 0.7071 & 0 \end{array} \right]$$

The optimal γ value for standard \mathcal{H}^∞ problem is $\gamma_{opt} = 1.2929$. Using the synthesis in [15], a stable \mathcal{H}^∞ controller is found at $\gamma_{min} = 1.36994$. When our method applied, we reached stable \mathcal{H}^∞ controller for $\gamma_{K,min} = 1.36957$. Although it seems slight improvement, our method is much more simpler with help of LMI problem formulation. Apart from standard problem solution (finding M_∞), the algorithm in [15] finds the stable \mathcal{H}^∞ controller by solving an additional \mathcal{H}^∞ problem.

Another common benchmark example (see [12] and its references) is to find a stable \mathcal{H}^∞ controller for the generalized plant described by (IV.8).

In [10], it is noted that for this problem, the sufficient condition in [7] is not satisfied for even large values of γ and the method is not applicable. As we can see from Table I, the performance of our method is better than the method in [10] except the last case. For all cases, the result of [12] is superior from all other methods. However, the controller order in [12] is 24 which is greater than our controller order, 16. To address this problem, in [12] a controller order reduction is performed, that results in lower order (e.g. 10th order for the case $\beta = 0.1$) stable controllers without significant loss of performance. Furthermore, the method in [12] involves solution of an additional \mathcal{H}^∞ problem which is complicated compared to our simple LMI formulation. If the algorithm in [12] fails, selection of a new parameter Q is suggested which is an ad-hoc procedure. Although the performance of the controller suggested in the present paper is slightly worse, it is numerically stable and easily formulated.

The following example is taken from [13]. Design a stable \mathcal{H}^∞ controller for the plant

$$P(s) = \frac{s^2 + 0.1s + 0.1}{(s - 0.1)(s - 1)(s^2 + 2s + 3)}.$$

For the mixed sensitivity minimization problem the weights are taken to be as in [13]. A comparison of the methods [10], [13], [14] and our method can be seen in Table II. There is a compromise between the methods. The performance of the method in [13] is worse than our method, but the order of our controller has twice order of the controller in [13]. Although the method in [14] gives better results than our method, the order of the controller in [14] is considerably

TABLE I
STABLE \mathcal{H}^∞ CONTROLLER DESIGN FOR (IV.8)

β	γ_{opt}	Gumussoy-Ozbay(GO)	[10]	[12]
		γ_{GO}	γ_{ZO}	γ_{CZ}
0.1	0.232	0.241	0.245	0.237
0.01	0.142	0.176	0.178	0.151
0.001	0.122	0.170	0.170	0.132

$$\begin{aligned}
z_1 &= \frac{0.03s^7 + 0.008s^6 + 0.19s^5 + 0.037s^4 + 0.36s^3 + 0.05s^2 + 0.18s + 0.015}{s^8 + 0.161s^7 + 6s^6 + 0.582s^5 + 9.984s^4 + 0.407s^3 + 3.9822s^2}(w_1 + u), \\
z_2 &= \beta u, \\
y &= w_2 + \frac{0.0064s^5 + 0.0024s^4 + 0.071s^3 + s^2 + 0.1045s + 1}{s^8 + 0.161s^7 + 6s^6 + 0.582s^5 + 9.984s^4 + 0.407s^3 + 3.9822s^2}(w_1 + u).
\end{aligned} \tag{IV.8}$$

higher than our controller order. However, this example clearly shows that our method is superior than [10].

As a remark, the method also gives very good results for single-input-single-output systems. The following SISO example is taken from [11]:

$$P(s) = \frac{(s+5)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-20)(s-30)},$$

$$W_1(s) = \frac{1}{s+1},$$

$$W_2(s) = 0.2,$$

the optimal \mathcal{H}^∞ problem is defined as

$$\gamma_{opt} = \inf_{K \text{ stabilizing } P} \left\| \begin{bmatrix} W_1(1+PK)^{-1} \\ W_2K(1+PK)^{-1} \end{bmatrix} \right\|_\infty$$

and the optimal performance for the given data is $\gamma_{opt} = 34.24$. A stable \mathcal{H}^∞ controller can be found for $\gamma = 42.51$ using the method of [11], whereas our method, which can be seen as a generalization of [11], gives a stable controller with $\gamma = 35.29$. In numerical simulations, we observed that when γ approaches to the minimum value satisfying sufficient conditions, the solutions of algebraic riccati equations of [11] become numerically ill-posed. However, the LMI based solution proposed here does not have such a problem. Same example is considered in [20] and stable \mathcal{H}^∞ controller found for $\gamma = 34.44$. The method in [20] is a two-stage algorithm with combination of genetic algorithm and quasi-Newton algorithm and gives slightly better performance than our method. The method finds stable \mathcal{H}^∞ controllers with a selection of low-order controller for free parameter Q . Since the example considered in the paper is for SISO case, it may be difficult to achieve good performance with low-order controller for MIMO case. Due to nonlinear optimization problem structure, the solution of the method may converge to local minima and in general, genetic algorithms give solution for longer time.

TABLE II
STABLE \mathcal{H}^∞ CONTROLLER DESIGN FOR EXAMPLE IN [13]

	Gumussoy-Ozbay	[10]	[13]	[14], Thm 7
γ_{min}	32.557	37.551	43.167	21.787
Order	$2n$	$2n$	n	$3n$

V. CONCLUDING REMARKS

In this paper, sufficient conditions for strong stabilization of MIMO systems are obtained and applied to stable \mathcal{H}^∞ controller design. Our conditions are based on linear matrix inequalities which can be easily solved by the LMI Toolbox of MATLAB. The method is very efficient from numerical point of view as demonstrated with examples. The benchmark examples show that the proposed method is a significant improvement over the existing techniques available in the literature. The exceptions to this claim are the methods of [12], [14], [20]. In [12], the controller design is based on ad-hoc search method, and both [13] and [14] result in higher order controllers than the one designed by our method. In [20], selection of low-order controller for Q gives good results for SISO structure of Q . However in MIMO structure, Q may not result in good performance.

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