On the Mixed Sensitivity Minimization for Systems with Infinitely Many Unstable Modes*

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Abstract

In this note we consider a class of linear time invariant systems with infinitely many unstable modes. By using the parameterization of all stabilizing controllers and a data transformation, we show that $\mathcal{H}^\infty$ controllers for such systems can be computed using the techniques developed earlier for infinite dimensional plants with finitely many unstable modes.

1 Introduction

It is well known that $\mathcal{H}^\infty$ controllers for linear time invariant systems with finitely many unstable modes can be determined by various methods, see e.g. [1, 2, 3, 4, 6, 8, 9, 10, 12, 13]. The main purpose of this note is to show that $\mathcal{H}^\infty$ controllers for systems with infinitely many unstable modes can be obtained by the same methods, using a simple data transformation. An example of such a plant is a high gain system with delayed feedback (see Section 3). Undamped flexible beam models, [7], may also be considered as a system with infinitely many unstable modes.

In earlier studies, e.g. [12], $\mathcal{H}^\infty$ controllers are computed for weighted sensitivity minimization involving plants in the form

$$P(s) = \frac{M_n(s)}{M_d(s)} N_o(s) \tag{1}$$

where $M_n(s)$ is inner and infinite dimensional, $M_d(s)$ is inner and finite dimensional, and $N_o(s)$ is the outer part of the plant, that is possibly infinite dimensional. In the weighted sensitivity minimization problem, the optimal controller achieves the minimum $\mathcal{H}^\infty$ cost, γ_{opt}, defined as

$$\gamma_{opt} = \inf_{C \text{ stabilizing } P} \left\| \begin{bmatrix} W_1(1+PC)^{-1} \\ W_2PC(1+PC)^{-1} \end{bmatrix} \right\|_\infty , \tag{2}$$

where W_1 and W_2 are given finite dimensional weights. Note that in the above formulation, the plant has finitely many unstable modes, because $M_d(s)$ is finite dimensional, whereas it may have infinitely many zeros in $M_n(s)$. In this note, by using duality, the mixed sensitivity minimization

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problem will be solved for plants with finitely many right half plane zeros and infinitely many unstable modes.

In Section 2 we show the link between the two problems and give the procedure to find optimal $\mathcal{H}^\infty$ controllers by using the procedure of the book by Foias *et al.*, [5]. In Section 3, a delay system example is given and the design steps for optimal controller are explained. Concluding remarks are made in Section 4.

2 Main Result

Assume that the plant to be controlled has infinitely many unstable modes, finitely many right half plane zeros and no direct transmission delay. Then, its transfer function is in the form $P = \frac{N}{M}$, where M is inner and infinite dimensional (it has infinitely many zeros in $\mathbb{C}_+$, that are unstable poles of P), $N = N_i\ N_o$ with N_i being inner finite dimensional, and N_o is the outer part of the plant, possibly infinite dimensional. For simplicity of the presentation we further assume that $N_o, N_o^{-1} \in \mathcal{H}^\infty$.

To use the controller parameterization of Smith, [11], we first solve for $X, Y \in \mathcal{H}^\infty$ satisfying

$$NX + MY = 1 \qquad \text{i.e.} \quad X(s) = \left(\frac{1 - M(s)Y(s)}{N_i(s)}\right) N_o^{-1}(s). \tag{3}$$

Let $z_1, ..., z_n$ be the zeros of $N_i(s)$ in $\mathbb{C}_+$, and again for simplicity assume that they are distinct. Then, there are finitely many interpolation conditions on $Y(s)$ for $X(s)$ to be stable, i.e.

$$Y(z_i) = \frac{1}{M(z_i)}.$$

Thus by Lagrange interpolation, we can find a finite dimensional $Y \in \mathcal{H}^\infty$ and infinite dimensional $X \in \mathcal{H}^\infty$ satisfying (3), and all controllers stabilizing the feedback system formed by the plant P and the controller C are parameterized as follows, [11],

$$C(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)} \quad \text{where } Q(s) \in \mathcal{H}^\infty \text{ and } (Y(s) - N(s)Q(s)) \neq 0. \tag{4}$$

Now we use the above parameterization in the sensitivity minimization problem. First note that,

$$(1 + P(s)C(s))^{-1} = M(s)(Y(s) - N(s)Q(s))$$

$$P(s)C(s)(1 + P(s)C(s))^{-1} = N(s)(X(s) + M(s)Q(s)). \tag{5}$$

Then,

$$\inf_{C \text{ stabilizing } P} \left\| \begin{bmatrix} W_1(1+PC)^{-1} \\ W_2PC(1+PC)^{-1} \end{bmatrix} \right\|_\infty = \inf_{Q \in \mathcal{H}^\infty \text{ and } Y - NQ \neq 0} \left\| \begin{bmatrix} W_1(Y - NQ) \\ W_2N(X + MQ) \end{bmatrix} \right\|_\infty \tag{6}$$

where W_1 and W_2 are given finite dimensional (rational) weights. From (3) equation, we have

$$\left\| \begin{bmatrix} W_1Y - W_1NQ \\ W_2N\left(\frac{1-MY}{N}\right) + W_2MNQ \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} W_1(Y - N_i(N_oQ)) \\ W_2(1 - M(Y - N_i(N_oQ))) \end{bmatrix} \right\|_\infty . \tag{7}$$

Thus, the $\mathcal{H}^\infty$ optimization problem reduces to

$$\gamma_{opt} = \inf_{Q_1 \in \mathcal{H}^\infty \text{ and } Y - N_i Q_1 \neq 0} \left\| \begin{bmatrix} W_1(Y - N_i Q_1) \\ W_2(1 - M(Y - N_i Q_1)) \end{bmatrix} \right\|_\infty \tag{8}$$

where $Q_1 = N_o Q$, and note that $W_1(s), W_2(s), N_i(s), Y(s)$ are rational functions, and $M(s)$ is inner infinite dimensional.

The problem defined in (8) has the same structure as the problem dealt in Chapter 5 of the book by Foias, Özbay and Tannenbaum (FÖT)), [5] (that is based on [10]), where skew Toeplitz approach has been used for computing $\mathcal{H}^\infty$ optimal controllers for infinite dimensional systems with finitely many right half plane poles. Our case is the dual of the problem solved in [5, 10], i.e., there are infinitely many poles in $\mathbb{C}_+$, but the number of zeros in $\mathbb{C}_+$ is finite. Thus by mapping the variables as shown below, we can use the results of [5, 10] to solve our problem:

$$W_1^{FÖT}(s) = W_2(s)$$

$$W_2^{FÖT}(s) = W_1(s)$$

$$X^{FÖT}(s) = Y(s)$$

$$Y^{FÖT}(s) = X(s)$$

$$M_d^{FÖT} = N_i(s)$$

$$M_n^{FÖT}(s) = M(s)$$

$$N_o^{FÖT}(s) = N_o^{-1}(s),$$

and the optimal controller, C, for the two block problem (6) is the inverse of optimal controller for the dual problem in [5], i.e., $\left(C_{opt}^{FÖT}\right)^{-1}$.

If we only consider the one block problem case, with $W_2 = 0$, then the minimization of

$$\|W_1(Y - N_i Q_1)\|_\infty$$

is simply a finite dimensional problem. On the other hand, minimizing

$$\|W_2(1 - M(Y - N_i Q_1))\|_\infty$$

is an infinite dimensional problem.

3 Example

In this section, we illustrate the computation of $\mathcal{H}^\infty$ controllers for systems with infinitely many right half plane poles. The example is a plant containing an internal delayed feedback:

$$P(s) = \frac{R(s)}{1 + e^{-hs} R(s)}$$

where $R(s) = k\left(\frac{s-a}{s+b}\right)$ with $k > 1$, $a > b > 0$ and $h > 0$. Note that the denominator term $(1+e^{-hs}R(s))$ has infinitely many zeros $\sigma_n \pm j\omega_n$, where $\sigma_n \to \sigma_o = \frac{\ln(k)}{h} > 0$, and $\omega_n \to (2n+1)\pi$, as $n \to \infty$. Clearly, $P(s)$ has only one right half plane zero at $s = a$.

The plant can be written as explained in Section 2,

$$P(s) = \frac{N_i(s)}{M(s)} N_o(s) \tag{9}$$

where

$$\begin{aligned} N_i(s) &= \left(\frac{s-a}{s+a}\right) \\ N_o(s) &= \frac{1}{1+\frac{(s-b)}{k(s+a)}e^{-hs}} \\ M(s) &= \frac{(s+b)+k(s-a)e^{-hs}}{(s-b)e^{-hs}+k(s+a)} \end{aligned}$$

It is clear that N_o is invertible in H^∞, because $\|\frac{s-b}{k(s+a)}\|_\infty < 1$. By the same argument, M is stable. To see that M is inner, we write it as

$$M(s) = \frac{m(s)+f(s)}{1+m(s)f(-s)}$$

with $m(s) = \left(\frac{s-a}{s+a}\right)e^{-hs}$, and $f(s) = \frac{s+b}{k(s+a)}$. Note that $m(s)$ is inner, $m(s)f(-s)$ is stable, and $M(s)M(-s) = 1$. Thus M is inner, and it has infinitely many zeros in the right half plane.

The optimal $\mathcal{H}^\infty$ controller can be designed for weighted sensitivity minimization problem in (2) where P is defined in (9) and weight functions are chosen as $W_1(s) = \rho$, $\rho > 0$ and $W_2(s) = \frac{1+\alpha s}{\beta+s}$, $\alpha > 0$, $\beta > 0$, $\alpha\beta < 1$. As explained before, this problem can be solved by the method in [5] after necessary assignments are done, $W_1^{FÖT}(s) = \frac{1+\alpha s}{\beta+s}$, $W_2^{FÖT}(s) = \rho$, $M_d^{FÖT} = \frac{s-a}{s+a}$,

$$M_n^{FÖT}(s) = \frac{(s+b)+k(s-a)e^{-hs}}{(s-b)e^{-hs}+k(s+a)}$$

$$N_o^{FÖT}(s) = \frac{(s-b)e^{-hs}+k(s+a)}{k(s+a)}.$$

We will briefly outline the procedure to find the optimal $\mathcal{H}^\infty$ controller.

1) Define the functions,

$$F_\gamma(s) = \gamma\left(\frac{\beta-s}{a_\gamma+b_\gamma s}\right), \qquad \omega_\gamma = \sqrt{\frac{1-\gamma^2\beta^2}{\gamma^2-\alpha^2}} \quad \text{for} \quad \gamma > 0$$

where $a_\gamma = \sqrt{1+\rho^2\beta^2-\rho^2\gamma^{-2}}$, and $b_\gamma = \sqrt{(1-\rho^2\gamma^{-2})\alpha^2+\rho^2}$.

2) Calculate the minimum singular value of the matrix,

$$M_\gamma = \begin{bmatrix} 1 & j\omega_\gamma & M(j\omega_\gamma)F_\gamma(j\omega_\gamma) & j\omega_\gamma M(j\omega_\gamma)F_\gamma(j\omega_\gamma) \\ 1 & a & M(a)F_\gamma(a) & aM(a)F_\gamma(a) \\ M(j\omega_\gamma)F_\gamma(j\omega_\gamma) & -j\omega_\gamma M(j\omega_\gamma)F_\gamma(j\omega_\gamma) & 1 & -j\omega_\gamma \\ M(a)F_\gamma(a) & -aM(a)F_\gamma(a) & 1 & -a \end{bmatrix}$$

for all values of $\gamma \in (\max\{\alpha, \frac{\rho}{\sqrt{1+\rho^2\beta^2}}\}, \frac{1}{\beta})$. The optimal gamma value, γ_{opt}, is the largest gamma which makes the matrix M_γ singular.

3) Find the eigenvector $l = [l_{10}, l_{11}, l_{20}, l_{21}]^T$ such that $M_{\gamma_{opt}} l = 0$.

4) The optimal $\mathcal{H}^\infty$ controller can be written as,

$$C_{opt}(s) = \frac{k_f + K_{2,FIR}(s)}{K_1(s)}$$

where k_f is constant, $K_1(s)$ is finite dimensional, and $K_{2,FIR}(s)$ is a filter whose impulse response is of finite duration

$$\begin{aligned} K_1(s) &= \frac{k(l_{21}s + l_{20})}{\gamma_{opt}(\beta + s)}, \\ k_f &= \left(\frac{k b_{\gamma_{opt}} l_{11} - \gamma_{opt} l_{21}}{\gamma_{opt}^2 - \alpha^2} \right), \\ K_{2,FIR}(s) &= A(s) + B(s)e^{-hs}, \\ k_f + A(s) &= \frac{k(s+a)(a_{\gamma_{opt}} + b_{\gamma_{opt}} s)(l_{11}s + l_{10}) + \gamma_{opt}(\beta - s)(l_{21}s + l_{20})(s+b)}{((1-\gamma_{opt}^2\beta^2) + (\gamma_{opt}^2 - \alpha^2)s^2)(s-a)}, \\ B(s) &= \frac{(s-b)(a_{\gamma_{opt}} + b_{\gamma_{opt}} s)(l_{11}s + l_{10}) + k\gamma_{opt}(\beta - s)(l_{21}s + l_{20})(s-a)}{((1-\gamma_{opt}^2\beta^2) + (\gamma_{opt}^2 - \alpha^2)s^2)(s-a)}. \end{aligned}$$

As a numerical example, if we choose the plant as $P(s) = \frac{2(\frac{s-3}{s+1})}{1+2(\frac{s-3}{s+1})e^{-0.5s}}$ and the weight functions as $W_1(s) = 0.5$, $W_2(s) = \frac{1+0.1s}{0.4+s}$, then the optimal $\mathcal{H}^\infty$ cost is $\gamma_{opt} = 0.5584$, and the corresponding controller is

$$C_{opt}(s) = \left(\frac{0.558s + 0.223}{2s + 3.725} \right) (1.477 + K_{2,FIR}(s))$$

where

$$K_{2,FIR}(s) = \frac{(2.0807s^2 - 6.3022s - 0.8264) - (0.6147s^3 - 0.7682s^2 - 5.2693s + 1.5870)e^{-0.5s}}{(0.3018s^3 - 0.9053s^2 + 0.9501s - 2.8504)}.$$

whose impulse response is of finite duration:

$$\mathcal{L}^{-1}(K_{2,FIR}(s)) = \begin{cases} -0.27e^{3t} + 7.16\cos(1.77t) + 0.36\sin(1.77t) - 2.037\delta(t-0.5) & 0 \le t \le 0.5 \\ 0 & t > 0.5 \end{cases}.$$

4 Conclusions

In this note we have considered $\mathcal{H}^\infty$ control of a class of systems with infinitely many right half plane poles. We have demonstrated that the problem can be solved by using the existing $\mathcal{H}^\infty$ control techniques for infinite dimensional systems with finitely many right half plane poles. An example from delay systems is given to illustrate the computational technique.

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