On Stable $\mathcal{H}^\infty$ Controllers for Time-Delay Systems*

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Abstract

In this paper, we study the stability of suboptimal $\mathcal{H}^\infty$ controllers for time-delay systems. The optimal $\mathcal{H}^\infty$ controller may have finitely or infinitely many unstable poles. A stable suboptimal $\mathcal{H}^\infty$ controller design procedure is given for each of these cases. The design methods are illustrated with examples.

1 Introduction

A strongly stabilizing controller is a stable controller in a stable feedback, [1]. In many practical applications, strongly stabilizing controllers are desired, see e.g. [2, 3, 4, 5, 6, 7, 8, 9] and their references. In these papers, direct design methods are given for $\mathcal{H}^\infty$ strong stabilization for finite dimensional plant case. The necessary and sufficient condition for strong stabilization, parity interlacing property, is shown in [10] for single input single output delay systems. A design method to find strongly stabilizing controller for single input single output systems with time delays is given [11] in which the stable controller is constructed by using the unit satisfying some interpolation conditions.

An indirect approach to design stable controller achieving a desired $\mathcal{H}^\infty$ performance level for time delay systems is given in [12]. This approach is based on stabilization of $\mathcal{H}^\infty$ controller by another $\mathcal{H}^\infty$ controller in the feedback loop. In [12], stabilization is achieved and the sensitivity deviation is minimized. There are two main drawbacks of this method. First, the solution of sensitivity deviation brings conservatism because of finite dimensional approximation of the infinite dimensional weight. Second, the stability of overall sensitivity function is not guaranteed. Also, overall system does not achieve the exact performance level, since the optimal $\mathcal{H}^\infty$ controller is perturbed by deviation.

*This work was supported in part by the National Science Foundation under grant ANI-0073725

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Our paper focuses on strong stabilization problem for infinite dimensional plants such that the stable controller achieves the pre-specified suboptimal $\mathcal{H}^\infty$ performance level. When the optimal controller is unstable (with infinitely or finitely many unstable poles), two methods are given based on a search algorithm to find a stable suboptimal controller. However, both methods are conservative. In other words, there may be a stable suboptimal controller achieving a smaller performance level, but the designed controller satisfies the desired overall $\mathcal{H}^\infty$ norm. The stability of optimal and suboptimal controller is discussed and necessity conditions are given.

It is known that a $\mathcal{H}^\infty$ controller for time-delay systems with finitely many unstable poles can be designed by the methods in [13, 14, 15, 16]. In general, weighted sensitivity problem results in an optimal $\mathcal{H}^\infty$ controller with infinitely unstable modes, [17, 18].

We assume that the plant is single input single output (SISO) and admits the representation as in [16],

$$P(s) = \frac{m_n(s)N_o(s)}{m_d(s)} \tag{1.1}$$

where $m_n(s) = e^{-hs}M(s)$, $h > 0$, and $M(s)$, $m_d(s)$ are finite dimensional, inner, and $N_o(s)$ is outer, possibly infinite dimensional. The optimal $\mathcal{H}^\infty$ controller, C_{opt}, stabilizes the feedback system and achieves the minimum $\mathcal{H}^\infty$ cost, γ_{opt}:

$$\gamma_{opt} = \left\| \begin{bmatrix} W_1(1+PC_{opt})^{-1} \\ W_2PC_{opt}(1+PC_{opt})^{-1} \end{bmatrix} \right\|_\infty = \inf_{C \text{ stabilizing } P} \left\| \begin{bmatrix} W_1(1+PC)^{-1} \\ W_2PC(1+PC)^{-1} \end{bmatrix} \right\|_\infty \tag{1.2}$$

where W_1 and W_2 are finite dimensional weights for the mixed sensitivity minimization problem.

In the next section, the structure of optimal and suboptimal $\mathcal{H}^\infty$ controllers will be summarized. The optimal controller with infinitely many unstable poles case is considered in Section 3. The conditions and a design method for stable suboptimal $\mathcal{H}^\infty$ controller is given in the same section. Similar work is done in Section 4 for the optimal controller with finitely many unstable poles. Examples related for these design methods are presented in Section 5, and concluding remarks can be found in Section 6.

2 Structure of $\mathcal{H}^\infty$ Controllers

Assume that the problem (1.2) satisfies $(W_2N_o), (W_2N_o)^{-1} \in \mathcal{H}^\infty$, then optimal $\mathcal{H}^\infty$ controller can be written as, [19],

$$C_{opt}(s) = E_{\gamma_{opt}}(s)m_d(s)\frac{N_o^{-1}(s)F_{\gamma_{opt}}(s)L(s)}{1+m_n(s)F_{\gamma_{opt}}(s)L(s)} \tag{2.3}$$

where $E_\gamma = \left(\frac{W_1(-s)W_1(s)}{\gamma^2} - 1\right)$, and for the definition of the other terms, let the right half plane zeros of $E_\gamma(s)$ be β_i, $i = 1, \ldots, n_1$, the right half plane poles of $P(s)$ be α_i, $i = 1, \ldots, l$

and that of $W_1(-s)$ be η_i $i = 1, \ldots, n_1$. Then, $F_\gamma(s) = G_\gamma(s) \prod_{i=1}^{n_1} \frac{s-\eta_i}{s+\eta_i}$ where

$$G_\gamma(s)G_\gamma(-s) = \left(1 - \left(\frac{W_2(-s)W_2(s)}{\gamma^2} - 1\right) E_\gamma\right)^{-1} \tag{2.4}$$

and $G_\gamma, G_\gamma^{-1} \in \mathcal{H}^\infty$, and $L(s) = \frac{L_2(s)}{L_1(s)}$, $L_1(s)$ and $L_2(s)$ are polynomials with degrees less than or equal to (n_1+l-1) and they are determined by the following interpolation conditions,

$$\begin{aligned} 0 &= L_1(\beta_i) + m_n(\beta_i)F_\gamma(\beta_i)L_2(\beta_i) \quad i = 1, \ldots, n_1 \\ 0 &= L_1(\alpha_i) + m_n(\alpha_i)F_\gamma(\alpha_i)L_2(\alpha_i) \quad i = 1, \ldots, l \\ 0 &= L_2(-\beta_i) + m_n(\beta_i)F_\gamma(\beta_i)L_1(-\beta_i) \quad i = 1, \ldots, n_1 \\ 0 &= L_2(-\alpha_i) + m_n(\alpha_i)F_\gamma(\alpha_i)L_1(-\alpha_i) \quad i = 1, \ldots, l. \end{aligned} \tag{2.5}$$

The optimal performance level, γ_{opt}, is the largest γ value such that spectral factorization (2.4) exists and interpolation conditions (2.5) are satisfied.

Similarly, the suboptimal controller achieving the performance level, ρ, can be defined as,

$$C_{subopt}(s) = E_\rho(s)m_d(s)\frac{N_o^{-1}(s)F_\rho(s)L_U(s)}{1 + m_n(s)F_\rho(s)L_U(s)} \tag{2.6}$$

where $\rho > \gamma_{opt}$ and $L_U(s) = \frac{L_{2U}(s)}{L_{1U}(s)} = \frac{L_2(s)+L_1(-s)U(s)}{L_1(s)+L_2(-s)U(s)}$ with $U \in \mathcal{H}^\infty$, $\|U\|_\infty \leq 1$. The polynomials, $L_1(s)$ and $L_2(s)$, have degrees less than or equal to $n_1 + l$. Same interpolation conditions are valid with ρ instead of γ. Moreover, there are two additional interpolation conditions for $L_1(s)$ and $L_2(s)$:

$$0 = L_2(-a) + (E_\rho(a) + 1)F_\rho(a)m_n(a)L_1(-a) \tag{2.7}$$

$$0 \neq L_1(-a) \tag{2.8}$$

where $a \in \mathbb{R}^+$ is arbitrary. The above terms and notations are the same as in [19].

Note that the unstable zeros of $E_{\gamma_{opt}}$ and m_d are always cancelled by the denominator in (2.3). Therefore, C_{opt} is stable if and only if the denominator in (2.3) has no unstable zeros except the unstable zeros of $E_{\gamma_{opt}}$ and m_d (multiplicities considered). Same conclusions are valid for the suboptimal case, C_{subopt} is stable provided that the denominator in (2.6) has unstable zeros only at the unstable zeros of E_ρ and m_d (again, multiplicities considered).

It is clear that the optimal, respectively suboptimal, controllers have infinitely many unstable poles if and only if there exists $\sigma_o > 0$ such that the following inequality holds

$$\lim_{\omega\to\infty} |F_{\gamma_{opt}}(\sigma_o + j\omega)L_{opt}(\sigma_o + j\omega)| > 1, \tag{2.9}$$

respectively,

$$\lim_{\omega\to\infty} |F_\rho(\sigma_o + j\omega)L_U(\sigma_o + j\omega)| > 1. \tag{2.10}$$

The controller may have infinitely many poles because of the delay term in the denominator. All the other terms are finite dimensional.

Even when the optimal controller has infinitely many unstable poles, a stable suboptimal controller may be found by proper selection of the free parameter $U(s)$. In Section 3 this case is discussed.

Note that the previous case covers one and two block cases (i.e., $W_2 = 0$ and $W_2 \neq 0$ respectively). When $F_{\gamma_{opt}}$ is strictly proper, then the optimal and suboptimal controllers may have only finitely many unstable poles. Existence of stable suboptimal $\mathcal{H}^\infty$ controllers and their design will be discussed in Section 4 for this case.

3 Stable suboptimal $\mathcal{H}^\infty$ controllers, when the optimal controller has infinitely many unstable poles

The following lemma gives the necessary condition for a suboptimal controller to have finitely many unstable poles.

Lemma 3.1. *Assume that the optimal controller has infinitely many unstable poles and $U(s)$ is finite dimensional, the suboptimal controller has finitely many unstable poles if and only if*

$$\lim_{\omega\to\infty} |F_\rho(j\omega)L_U(j\omega)| \leq 1 \tag{3.11}$$

Proof Assume that the suboptimal controller has infinitely many unstable poles, then the equation

$$1 + e^{-h(\sigma+j\omega)}M(\sigma+j\omega)F_\rho(\sigma+j\omega)L_U(\sigma+j\omega) = 0$$

has infinitely many zeros in the right half plane,i.e., there exists $\sigma = \sigma_o > 0$ and for sufficiently large ω,

$$1 + e^{-h(\sigma_o+j\omega)} \lim_{\omega\to\infty} \left(F_\rho(\sigma_o+j\omega)L_U(\sigma_o+j\omega)\right) = 0 \tag{3.12}$$

will have infinitely many zeros. Since F_ρ and L_U are finite dimensional,

$$\begin{aligned}
\lim_{\omega\to\infty} F_\rho(j\omega) &= \lim_{\omega\to\infty} F_\rho(\sigma+j\omega) \\
\lim_{\omega\to\infty} L_U(j\omega) &= \lim_{\omega\to\infty} L_U(\sigma+j\omega) \quad \forall\, \sigma > 0.
\end{aligned}$$

By using this fact, we can rewrite (3.12) as,

$$1 + e^{-h(\sigma_o+j\omega)} \lim_{\omega\to\infty} \left(F_\rho(j\omega)L_U(j\omega)\right) = 0 \tag{3.13}$$

which implies that in order to have infinitely many zeros, the condition in lemma should satisfied. Conversely, a similar idea can be used to show that (3.11) implies finitely many unstable poles. □

Note that this lemma is valid not only for only finite dimensional $U(s)$ term, but also for any $U \in \mathcal{H}^\infty$, $\|U\|_\infty \le 1$ provided that

$$\lim_{\omega\to\infty} U(j\omega) = \lim_{\omega\to\infty} U(\sigma + j\omega) = u_\infty, \quad \forall\, \sigma > 0. \tag{3.14}$$

is satisfied where $u_\infty \in \mathbb{R}$. Also, we can find conditions on U which guarantees finitely many unstable poles by using the lemma.

Assume that $U(s)$ is finite dimensional and bi-proper, and define

$$\begin{aligned} f_\infty &= \lim_{\omega\to\infty} |F_\rho(j\omega)| > 1 \\ u_\infty &= \lim_{\omega\to\infty} U(j\omega) \\ k &= \lim_{\omega\to\infty} \frac{L_2(j\omega)}{L_1(j\omega)} \end{aligned}$$

Lemma 3.2. *The suboptimal controller has finitely many unstable poles if and only if the following inequalities hold:*

$$|k| \le \frac{1}{f_\infty}, \quad |u_\infty| \le \frac{1 - f_\infty|k|}{f_\infty - |k|} \tag{3.15}$$

when $(n_1 + l)$ *is odd (even) and* $ku_\infty < 0, (ku_\infty > 0)$*, and*

$$|k| < 1, \quad \frac{f_\infty|k| - 1}{f_\infty - |k|} < |u_\infty| < \frac{f_\infty|k| + 1}{f_\infty + |k|} \tag{3.16}$$

when $(n_1 + l)$ *is odd (even) and* $ku_\infty > 0, (ku_\infty < 0)$*.*

Proof By using Lemma 3.1, when $(n_1 + l)$ is odd (even) and $ku_\infty < 0, (ku_\infty > 0)$, we can re-write (3.11) as

$$f_\infty \frac{|k| + |u_\infty|}{1 + |k||u_\infty|} \le 1.$$

After algebraic manipulations and using $f_\infty > 1$, we can show that (3.15) satisfies this condition. Similarly, when (n_1+l) is odd (even) and $ku_\infty > 0, (ku_\infty < 0)$, (3.11) is equivalent to

$$f_\infty \left| \frac{|k| - |u_\infty|}{1 - |k||u_\infty|} \right| \le 1.$$

, and (3.16) satisfies this condition. □

Note that u_∞ is a design parameter and the range can be determined, by given f_∞ and k.

Theorem 3.1. *Assume that the optimal and central suboptimal controller (when* $U = 0$*) has infinitely many unstable poles, if there exists* $U \in \mathcal{H}^\infty$*,* $\|U\|_\infty < 1$ *such that* L_{1U} *has no* $\mathbb{C}_+$ *zeros and* $|L_U(j\omega)F_\rho(j\omega)| \le 1$*,* $\forall\, \omega \in [0, \infty)$*, then the suboptimal controller is stable.*

Proof Assume that there exists U satisfying the conditions of the theorem. By maximum modulus theorem,

$$|1 + e^{-hs_o} M(s_o) F_\rho(s_o) L_U(s_o)| > 1 - e^{-h\sigma} |F_\rho(j\omega) L_U(j\omega)| > 0,$$

therefore, there is no unstable zero, $s_o = \sigma + j\omega$ with $\sigma > 0$. Since, all imaginary axis zeros are cancelled by E_ρ, the suboptimal controller has no unstable poles. □

The theorem has two disadvantages. First, there is no information for calculation of an appropriate parameter, U. Second, the inequality brings conservatism and there may exist stable suboptimal controllers even when the condition is violated. It is difficult to reveal the first problem, therefore it is better to use first order bi-proper function for U. For the second problem, define ω_{max} and η_{max} as,

$$\begin{aligned} \omega_{max} &= \max_{|L_U(j\omega) F_\rho(j\omega)|=1} \omega, \\ \eta_{max} &= \max_{\omega \in [0,\infty)} |L_U(j\omega) F_\rho(j\omega)|. \end{aligned}$$

It is important to design ω_{max} and η_{max} as small as possible by the choice of U. Otherwise, at high frequencies the delay term will generate unstable zeros when ω_{max} is large. Similarly, when η_{max} is large, although ω_{max} is small, it may cause unstable zeros. The design method given below searches for a first order U, and it is based on the above ideas. An example will be given in Section 5.

Algorithm
Define $U(s) = u_\infty \left(\frac{u_z + s}{u_p + s} \right)$ such that $u_\infty, u_p, u_z \in \mathbb{R}$, $|u_\infty| < 1$, $u_p > 0$ and $u_p \geq u_\infty |u_z|$,

1) Fix $\rho > \gamma_{opt}$,

2) Obtain f_∞ and k from the central suboptimal controller,

3) Calculate admissible values of u_∞ by using Lemma (3.2),

4) Search admissible values for (u_∞, u_p, u_z) such that $L_{1U}(s)$ is stable,

5) Find the minimum ω_{max} and η_{max} for all admissible (u_∞, u_p, u_z).

6) Check in the region $D = \{s = \sigma + j\omega, \sigma \geq 0 : |e^{-hs} M(s) F_\rho(s) L_U(s)| > 1\}$ whether $1 + e^{-hs} M(s) F_\rho(s) L_U(s)$ has no $\mathbb{C}_+$ zeros except unstable zeros of E_ρ and m_d.

When the central suboptimal controller has infinitely many unstable poles, it is not possible to obtain a stable suboptimal controller by a choice of U as strictly proper or inner function. Once we find a U from the above algorithm, the resulting stable suboptimal $\mathcal{H}^\infty$ controller can be represented as cascade and feedback connections of finite dimensional terms and a finite impulse response filter that does not have unstable pole-zero cancellations in the controller, as explained in [20].

4 Stable suboptimal $\mathcal{H}^\infty$ controllers, when the optimal controller has finitely many unstable poles

In this section, we will derive the conditions for the $\mathcal{H}^\infty$ controllers to have finitely many unstable poles. A sufficient condition for the existence of stable suboptimal $\mathcal{H}^\infty$ controllers is given, and a design method will be derived.

The optimal and suboptimal controllers have infinitely many unstable poles, when $F_{\gamma_{opt}} L_{opt}$ and $F_\rho L_U$ has magnitude greater than one as $\omega \to \infty$. It is not difficult to see that controllers will have finitely many unstable poles if $F_{\gamma_{opt}}$ and F_ρ are strictly proper. Since, these terms decrease as $\omega \to \infty$ and delay term decays as σ increases, only finitely many unstable poles may appear. Clearly, there may be $\mathcal{H}^\infty$ controllers (depending on parameter values) with finitely many poles while $F_{\gamma_{opt}}$ and F_ρ are bi-proper. However, it is important to find the sufficient conditions when they are strictly proper, which results in controllers with finitely many unstable poles regardless of parameters.

Lemma 4.1. *The $\mathcal{H}^\infty$ controller has finitely many unstable poles if the plant is strictly proper and W_1 is proper (in the sensitivity minimization problem) and, W_1 is proper and W_2 is improper (in the mixed sensitivity minimization problem).*

Proof Transfer function $F(s)$can be written as ratio of two polynomials, N_F and D_F, with degrees m and n respectively. We can define relative degree function, ϕ, as

$$\phi(F(s)) = \phi\left(\frac{N_F(s)}{D_F(s)}\right) = n - m.$$

Note that $\phi(F_1(s)F_2(s)) = \phi(F_1(s)) + \phi(F_2(s))$ and $\phi(F(s)F(-s)) = 2\phi(F(s))$.

The optimal controller has finitely many unstable poles if $F_{\gamma_{opt}}$ is strictly proper, i.e. $\phi(F_{\gamma_{opt}}(s)) > 0$. To show this, we can write by using definition of $F_{\gamma_{opt}}$ and (2.4),

$$\begin{aligned}
\phi(F_{\gamma_{opt}}(s)) &= \phi(G_{\gamma_{opt}}(s)), \\
&= \frac{1}{2}\,\phi(\big(W_1(s)W_1(-s) + W_2(s)W_2(-s) - \gamma_{opt}^{-2} W_1(s)W_1(-s)W_2(s)W_2(-s)\big)^{-1}), \\
&= -\frac{1}{2}\,\phi(\big(W_1(s)W_1(-s) + W_2(s)W_2(-s) - \gamma_{opt}^{-2} W_1(s)W_1(-s)W_2(s)W_2(-s)\big)), \\
&= -\frac{1}{2}\,\min\{\phi(W_1(s)W_1(-s)), \phi(W_2(s)W_2(-s)), \phi(W_1(s)W_1(-s)W_2(s)W_2(-s))\}, \\
&= -\min\{\phi(W_1(s)), \phi(W_2(s)), \phi(W_1(s)) + \phi(W_2(s))\}.
\end{aligned}$$

Strictly properness of $F_{\gamma_{opt}}$ implies,

$$\min\{\phi(W_1(s)), \phi(W_2(s)), \phi(W_1(s)) + \phi(W_2(s))\} < 0. \tag{4.17}$$

We know that $\phi(W_1(s)) \geq 0$ and $\phi(W_2(s)) \leq 0$, [19]. Therefore, the inequality (4.17) is satisfied if and only if $\phi(W_1(s)) \geq 0$ and $\phi(W_2(s)) < 0$ are valid which means that $W_1(s)$ is proper and $W_2(s)$ is improper. Since we have $(W_2 N_o)^{-1} \in \mathcal{RH}^\infty$ [19], we can conclude that the plant is strictly proper. Same proof is valid for the suboptimal case. $\square$

We know that the suboptimal controllers are written as (2.6),

$$C_{subopt}(s) = E_\rho(s) m_d(s) \frac{N_o^{-1}(s) F_\rho(s) L_U(s)}{1 + m_n(s) F_\rho(s) L_U(s)}$$

we can rewrite the suboptimal controllers as,

$$C_{subopt}(s) = \left(\frac{N_o^{-1}(s) F_\rho(s)}{dE_\rho(s) dm_d(s)} \right) \left(\frac{L_2(s) + L_1(-s) m_n(s) F_\rho(s)}{P_1(s) + P_2(s) U(s)} \right)$$

where

$$\begin{aligned} P_1(s) &= \frac{L_1(s) + L_2(s) m_n(s) F_\rho(s)}{dE_\rho(s) dm_d(s)}, \\ P_2(s) &= \frac{L_2(-s) + L_1(-s) m_n(s) F_\rho(s)}{nE_\rho(s) nm_d(s)}, \end{aligned}$$

and nE_ρ, dE_ρ and nm_d, dm_d are numerator and denominator of E_ρ and m_d respectively. Denominators of P_1 and P_2 are cancelled by numerators.

Note that unstable poles of C_{subopt} are the zeros of $P_1 + P_2 U$. If there exists a $U \in \mathcal{RH}^\infty$ with $\|U\|_\infty < 1$, such that $P_1 + P_2 U$ has no unstable zeros, then the corresponding suboptimal controller is stable.

Assume that F_ρ is strictly proper which implies P_1 and P_2 has finitely many unstable zeros. The suboptimal controller is stable if and only if $S_U := (1 + \tilde{P} U)^{-1}$ is stable where $\tilde{P} = \frac{P_2}{P_1}$. Note that since P_1 and P_2 has finitely many unstable zeros, we can write $\tilde{P}$ as,

$$\tilde{P} = \frac{\tilde{M}}{\tilde{M}_d} \tilde{N}_o$$

where $\tilde{M}$ and $\tilde{M}_d$ are inner, finite dimensional and $\tilde{N}_o$ is outer and infinite dimensional. Finding stable S_U with $U \in \mathcal{H}^\infty$ is a sensitivity minimization problem with stable controller which is considered in [6]. However, in our case, U has a norm restriction as $\|U\|_\infty \leq 1$ and we can write U as,

$$U(s) = \left(\frac{1 - S_U(s)}{S_U(s)} \right) \left(\frac{P_1(s)}{P_2(s)} \right).$$

Define μ_{opt} as,

$$\mu_{opt} = \inf_{U \in \mathcal{H}^\infty} \|S_U\|_\infty = \inf_{U \in \mathcal{H}^\infty} \|(1 + \tilde{P} U)^{-1}\|_\infty.$$

If we fix μ as $\mu > \mu_{opt}$, then there exists a free parameter Q ($Q \in \mathcal{H}^\infty$ and $\|Q\|_\infty \leq 1$) which parameterizes all functions stabilizing S_U and achieving performance level μ. We will show that the sensitivity function achieving performance level μ as $S_{U,\mu}(Q)$.

Lemma 4.2. *Assume that W_1 and W_2 are proper and improper respectively. If there exists $\mu_o > \mu_{opt}$ and Q_o with $Q_o \in \mathcal{H}^\infty$ and $\|Q_o\|_\infty \leq 1$ satisfying*

$$\left| \left(\frac{1 - S_{U,\mu_o}(Q_o(j\omega))}{S_{U,\mu_o}(Q_o(j\omega))} \right) \left(\frac{P_1(j\omega)}{P_2(j\omega)} \right) \right| \leq 1, \tag{4.18}$$

then the suboptimal controller, C_{subopt}, achieves the performance level ρ by selecting the parameter U as,

$$U(s) = \left(\frac{1 - S_{U,\mu_o}(Q_o(s))((s)}{S_{U,\mu_o}(Q_o(s))} \right) \left(\frac{P_1(s)}{P_2(s)} \right) \tag{4.19}$$

Proof The result of theorem is immediate. Since Q_o satisfies the norm condition of U and makes $S_{U,\mu}(Q_o)$ stable, the suboptimal controller has no right half plane poles by selection of U as shown in theorem. □

A stable suboptimal controller can be designed by finding Q_o for μ_o. By using a search algorithm, we can find Q_o satisfying the norm condition for U. Instead of finding U resulting stable suboptimal controller, the problem is converted finding Q_o satisfying the norm condition. First problem needs to check whether a quasi-polynomial has unstable zeros. However, by using the theorem, this problem reduced into searching stable function with infinity norm less than one and satisfying norm condition for U. Conservatively, the search algorithm for Q_o can be done for first order bi-proper functions such that $Q_o(s) = u_\infty \left(\frac{s+z_u}{s+p_u} \right)$ where $p_u > 0$, $z_u \in \mathbb{R}$, and $|u_\infty| \leq \max\{1, \frac{p_u}{|z_u|}\}$. The algorithm for this approach is explained below.

Algorithm

Assume that the optimal and central suboptimal controllers have finitely many unstable poles. We can design a stable suboptimal $\mathcal{H}^\infty$ controller by using the following algorithm:

1) Fix $\rho > \gamma_{opt}$,

2) Obtain P_1 and P_2. If P_1 has no unstable zeros, then the suboptimal controller is stable for $U = 0$. If not, go to step 3.

3) Define the right half plane zeros of P_1 and P_2 as p_i and s_i respectively. Note that these are right half plane zeros of $\tilde{M}_d(s)$ and $\tilde{M}(s)$ respectively. Calculate $w_i = \frac{1}{\tilde{M}_d(s_i)}$ and $z_i = \frac{s_i - a}{s+a}$ where $a > 0$.

4) Search for minimum μ which makes the Pick matrix positive semi-definite,

$$Q_P\{\mu\}_{(i,k)} = \left(\frac{-\ln \frac{w_i}{\mu} - \ln \frac{\bar{w}_k}{\mu} + j2\pi(n_k - n_i)}{1 - z_i \bar{z}_k} \right) \tag{4.20}$$

where $n_{[.]}$ is integer. Note that most of the integers will not result in positive semi-definite Pick matrix. Therefore, for each integer set, we can find the smallest μ and μ_{opt} will be the minimum of these vales. For details, see [6].

5) After the integer set and μ_{opt} is found, the function $g(z) \in \mathcal{H}^\infty$ can be obtained satisfying interpolation conditions,

$$g(z_i) = -\ln \frac{w_i}{\mu_{opt}} - j2\pi n_i \tag{4.21}$$

by Nevanlinna-Pick interpolation approach [19],[21]. Then, we can write $S_U(s) = \mu_{opt}\tilde{M}_d(s)e^{-G(s)}$ where $G(s) = g(\frac{s-a}{s+a})$ and obtain $U(s)$. Check the norm condition $\|U\|_\infty \leq 1$. If it is satisfied, then, $U(s)$ results in stable suboptimal controller achieving performance level ρ. If not, go to next step.

6) Increase μ such that $\mu > \mu_{opt}$. For all possible integer set, obtain $g(z) \in \mathcal{H}^\infty$ with interpolation conditions,

$$g(z_i) = -\ln \frac{w_i}{\mu} - j2\pi n_i. \tag{4.22}$$

Note that since $g(z)$ has a free parameter $q(z)$ ($q \in \mathcal{H}^\infty$ and $\|q\|_\infty \leq 1$), we can write the function as $g(z, q)$. Then, search for parameters (u_∞,z_u,p_u) satisfying

$$\left| \left(\frac{1 - \mu\tilde{M}_d(j\omega)e^{-G(j\omega,Q)}}{\mu\tilde{M}_d(j\omega)e^{-G(j\omega,Q)}} \right) \left(\frac{P_1(j\omega)}{P_2(j\omega)} \right) \right| \leq 1, \quad \forall\omega \in [0, \infty) \tag{4.23}$$

where $G(s, Q(s)) = g(\frac{s-a}{s+a}, q(\frac{s-a}{s+a}))$ and $Q(s) = u_\infty \left(\frac{s+z_u}{s+p_u}\right)$ as defined before. If one of the parameter set satisfies the inequality, then $Q_o = u_{\infty,o}\left(\frac{s+z_{u,o}}{s+p_{u,o}}\right)$ and corresponding U results in a stable suboptimal $\mathcal{H}^\infty$ controller. If no parameter set satisfies the inequality, go to step 6, and repeat the procedure for sufficiently high μ, until a pre-specified maximum is reached, in which case go next step.

7) Increase ρ, go to step 2, if a maximum pre-specified ρ is reached, stop. This method fails to provide a stable $\mathcal{H}^\infty$ controller.

An illustrative example is presented in Section 5.

5 Examples

Two examples will be given in this section. In the first example, the optimal and central suboptimal controllers have infinitely many unstable poles; by using the design method, we show that there exists a stable suboptimal controller even the magnitude condition ($|L_U(j\omega)F_\rho(j\omega)| \leq 1$) is violated for low frequencies. In other words, the example illustrates that the conditions in (3.1) are sufficient.

The second example explains the design method for stable suboptimal $\mathcal{H}^\infty$ controller whose central controller is unstable with finitely many unstable poles and implements the algorithm step by step as mentioned in section 4.

5.1 Example

Let $P(s) = e^{-0.1s}\left(\frac{s-1}{s+1}\right)$ and choose $W_1(s) = \frac{1+0.6s}{s+1}$ and $W_2 = 0$ (one-block problem). Using Skew-Teoplitz approach in [19], the minimum $\mathcal{H}^\infty$ value, γ_{opt}, is 0.8108. The optimal controller has infinitely many unstable poles converging to $s = 3.0109 \pm j\frac{(2k+1)\pi}{h}$ as $k \to \infty$. If central suboptimal controller ($U = 0$) is calculated for $\rho = 0.814$, it has infinitely many unstable poles converging to $s = 2.445 \pm j\frac{(2k+1)\pi}{h}$ as $k \to \infty$. The suboptimal controllers can be represented as,

$$C_{subopt}(s) = E_\rho(s)\frac{F_\rho(s)L_U(s)}{1 + m_n(s)F_\rho(s)L_U(s)} \tag{5.24}$$

where

$$\begin{aligned}
m_n(s) &= e^{-0.1s}\left(\frac{s-1}{s+1}\right), \\
E_\rho(s) &= \frac{0.3374 + 0.3026s^2}{0.6626(1-s^2)}, \\
F_\rho(s) &= 0.814\left(\frac{1-s}{1+0.6s}\right), \\
L_U(s) &= \frac{L_{2U}(s)}{L_{1U}(s)} = \frac{L_2(s) + L_1(-s)U(s)}{L_1(s) + L_2(-s)U(s)}, \\
L_2(s) &= -(0.9413s + 1.8716), \\
L_1(s) &= (s + 1.8373).
\end{aligned}$$

We will use the design method of the Section 3 to find a stable suboptimal controller by search for U. The central suboptimal controller ($U = 0$) has infinitely many unstable poles as mentioned before. The algorithm is tried for $u_z = u_p = 0$ case, i.e., $U(s) = u_\infty$.

1) Fix $\rho = 0.814 > \gamma_{opt} = 0.8108$,

2) $k = -0.9413$ and $f_\infty = 1.3567$ are calculated.

3) $n_1 = 1$, $l = 0$, $n_1 + l$ is odd and $|k| > \frac{1}{f_\infty}$. By using Lemma (3.2), the admissible values for u_∞ are $-0.9909 < u_\infty < -0.6668$.

4) $L_{1U}(s)$ is stable for $u_\infty \in [-1, 0.98]$.

5) Overall admissible values for U are $u_\infty \in [-0.9909, -0.6668]$. The values of ω_{max} and η_{max} for all admissible u_∞ range can be seen in Figure 1. Since η_{max} values do not vary much, the minimum value of ω_{max} determines the optimal u_∞ value as $\omega_{max} = 19.458$ at $u_\infty = -0.813$.

6) Figure 2 shows the plot of $Z(s) = |1 + e^{-hs}M(s)F_\rho(s)L_U(s)|$ in the right half plane. The function has only right half plane zero at $s = \pm 1.056j$, which is right half zeros of $E_\rho(s)$. Note that, only one part of right plane is graphed since the other half is same.

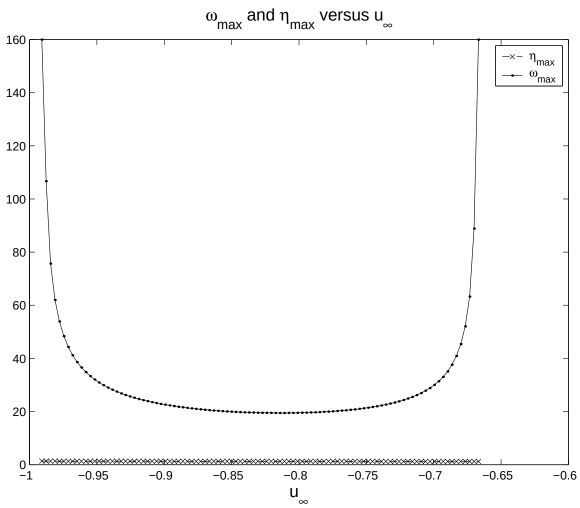

Figure 1: w_{max} and η_{max} versus u_∞

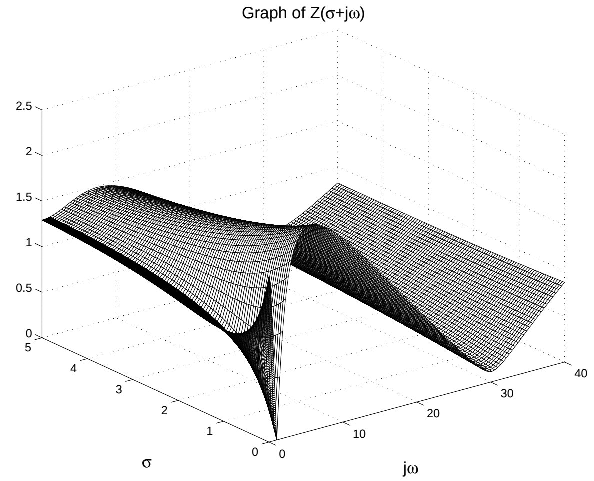

Figure 2: $Z(s)$ plot for right half plane

Therefore, we can conclude that suboptimal controller is stable for $U(s) = -0.813$ and achieves the $\mathcal{H}^\infty$ norm $\rho = 0.814$.

5.2 Example

For given plant $P(s) = e^{-3s}$ and weight functions $W_1(s) = \left(\frac{2.24+s}{1+s}\right)$ and $W_2(s) = 0.5(2.24+s)$, we can find the optimal performance level as $\gamma_{opt} = 1.9452$. The corresponding optimal $\mathcal{H}^\infty$ controller can be written as,

$$C_{opt}(s) = E_{\gamma_{opt}}(s)\frac{F_{\gamma_{opt}}(s)L_{opt}(s)}{1 + m_n(s)F_{\gamma_{opt}}(s)L_{opt}(s)} \tag{5.25}$$

where

$$\begin{aligned} m_n(s) &= e^{-3s}, \\ E_{\gamma_{opt}}(s) &= \frac{1.2162 + 2.7838s^2}{3.7838(1-s^2)}, \\ F_\rho(s) &= 5.5119\frac{(1-s)}{(2.24+s)^2}, \\ L_{opt}(s) &= 1. \end{aligned}$$

The optimal controller has unstable poles at $s = 0.0292 \pm 2.2354j$. Note that since W_1 and W_2 are proper and improper respectively, all $\mathcal{H}^\infty$ controllers will have finitely many unstable poles by Theorem 4.2. Therefore we can apply the algorithm in section 4.

1) Fix $\rho = 1.9454 > \gamma_{opt} = 1.9452$,

2) The suboptimal controllers can be written as,

$$C_{subopt}(s) = E_\rho(s)\frac{F_\rho(s)L_U(s)}{1 + m_n(s)F_\rho(s)L_U(s)} \tag{5.26}$$

where

$$
\begin{aligned}
m_n(s) &= e^{-3s}, \\
E_\rho(s) &= \frac{1.2154 + 2.7846s^2}{3.7846(1-s^2)}, \\
F_\rho(s) &= 5.5115\frac{(1-s)}{(2.24+s)^2}, \\
L_U(s) &= \frac{L_{2U}(s)}{L_{1U}(s)} = \frac{L_2(s) + L_1(-s)U(s)}{L_1(s) + L_2(-s)U(s)}, \\
L_2(s) &= (2.9837 + 0.9946s), \\
L_1(s) &= (2.9829 + s),
\end{aligned}
$$

and U is free parameter such that $U \in \mathcal{H}^\infty$, $\|U\|_\infty \leq 1$. We can write P_1 and P_2 as,

$$
\begin{aligned}
P_1(s) &= \frac{L_1(s) + m_n(s)F_\rho(s)L_2(s)}{nE_\rho(s)}, \\
&= \frac{(2.9829+s)(2.24+s)^2 + 5.5115(1-s)(2.9837+0.9946s)e^{-3s}}{(1.2154+2.7846s^2)(2.24+s)^2}, \\
P_2(s) &= \frac{L_2(-s) + m_n(s)F_\rho(s)L_1(-s)}{nE_\rho(s)}, \\
&= \frac{(2.9837-0.9946s)(2.24+s)^2 + 5.5115(1-s)(2.9829-s)e^{-3s}}{(1.2154+2.7846s^2)(2.24+s)^2}.
\end{aligned}
$$

Note that P_1 and P_2 has unstable zeros at $0.0287 \pm 2.2346j$ and $0.0297 \pm 2.2346j$ respectively. Therefore, the central controller ($U = 0$) for the chosen performance level, $\rho = 1.9458$, is unstable.

3) Define the following variables and functions as,

$$
\begin{aligned}
p_i &= 0.0287 \pm 2.2346j, \quad i = 1,2, \\
s_i &= 0.0297 \pm 2.2346j, \quad i = 1,2, \\
\tilde{M}_d(s) &= \frac{(s-p_1)(s-p_2)}{(s+p_1)(s+p_2)} = \frac{s^2 - 0.0574s + 4.9943}{s^2 + 0.0574s + 4.9943}, \\
w_i &= \frac{1}{\tilde{M}_d(s_i)} = 58.4002 \mp 0.7501j, \quad i = 1,2, \\
z_i &= \frac{s_i - 1}{s_i + 1} = 0.6598 \pm 0.7383i
\end{aligned}
$$

where conformal mapping parameter, a, is chosen as $a = 1$.

4) In order to find the minimum μ resulting in positive semi-definite Pick matrix,

$$
Q_P\{\mu\} = \begin{pmatrix} \frac{-8.1348 + 2\ln\mu}{0.0196} & \frac{(-8.1348+0.0257j)+2\ln\mu + j2\pi(n_2-n_1)}{1.1097-0.9742j} \\ \frac{(-8.1348-0.0257j)+2\ln\mu + j2\pi(n_1-n_2)}{1.1097-0.9742j} & \frac{-8.1348+2\ln\mu}{0.0196} \end{pmatrix}, \quad (5.27)
$$

we will find the minimum μ for all possible integer pairs (n_1, n_2). It is not difficult to do this search since many integer pairs do not result in positive semi-definite Pick matrix. For each integer pair, we can find the minimum μ, μ_{min}, and then μ_{opt} will be smallest of all μ_{min}. Note that since Pick matrix depends on difference of integers, we can normalize the search by taking $n_1 = 0$. In Figure 3,we can see the minimum μ values for integers, n_2. The minimum of all μ_{min} values is $\mu_{opt} = 58.4167$.

5) The calculation of $U(s)$ for μ_{opt} is omitted. It does not satisfy the norm condition $\|U\|_\infty \leq 1$.

6) Fix $\mu = 64$ and $n_1 = n_2 = 0$. The interpolation conditions for $g(z)$ can be written as,

$$g(z_i) = 0.0915 \pm 0.0128j, \quad i = 1, 2. \tag{5.28}$$

By Nevanlinna-Pick approach, (see e.g.[19]),

$$g(z,q) = \frac{(1.0878z^2 - 1.3782z + 0.9804)q(z) + (0.0724z - 0.1054)}{(0.9804z^2 - 1.3782z + 1.0878) + (0.0724z - 0.1054z^2)q(z)} \tag{5.29}$$

where $q(z)$ is a parameterization term such that $q \in \mathcal{H}^\infty$ and $\|q\|_\infty \leq 1$. The search algorithm tries to find q_o satisfying the norm condition

$$\left| \left(\frac{1 - \mu \tilde{M}_d(j\omega) e^{-G(j\omega,Q)}}{\mu \tilde{M}_d(j\omega) e^{-G(j\omega,Q)}} \right) \left(\frac{P_1(j\omega)}{P_2(j\omega)} \right) \right| \leq 1, \quad \forall \omega \in [0, \infty) \tag{5.30}$$

where

$$\begin{aligned} G(s, Q(s)) &= g\left(\frac{s-1}{s+1}, q\left(\frac{s-1}{s+1} \right) \right), \\ &= \frac{(0.69s^2 - 0.2148s + 3.4464)Q(s) - (0.0330s^2 + 0.3556s + 0.0330)}{(0.69s^2 + 0.2148s + 3.4464) - (0.0330s^2 - 0.3556s + 0.0330)Q(s)} \end{aligned}$$

and $Q \in \mathcal{H}^\infty$, $\|Q\|_\infty \leq 1$. We will search for Q satisfying the norm condition (5.30) in the form of $Q(s) = u_\infty$ with $|u_\infty| \leq 1$. Note that we choose $z_u = p_u = 0$ and all functions in norm condition, P_1, P_2, $\tilde{M}_d$, are defined before. After search is done, the condition (5.30) is satisfied for $u_\infty = 0.323$. The magnitude of $U(j\omega)$ is smaller than one for all frequency values as seen in Figure. (i.e., $\|U\|_\infty = 0.9924$). As a result, the suboptimal $\mathcal{H}^\infty$ controller achieving the performance level, $\rho = 1.9454$, is stable with selection of parameter U as,

$$U(s) = \left(0.0156 \left(\frac{s^2 + 0.0574s + 4.9943}{s^2 - 0.0574s + 4.9943} \right) e^{\left(\frac{0.1899s^2 - 0.4250s + 1.0802}{0.6793s^2 + 0.3297s + 3.4357} \right)} - 1 \right) \left(\frac{P_1(s)}{P_2(s)} \right). \tag{5.31}$$

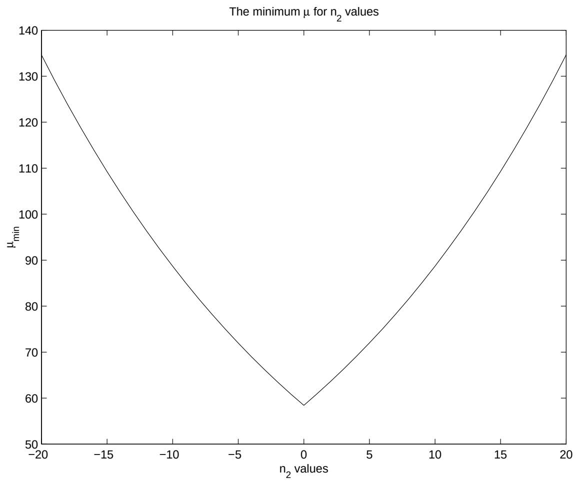

Figure 3: μ_{min} versus n_2

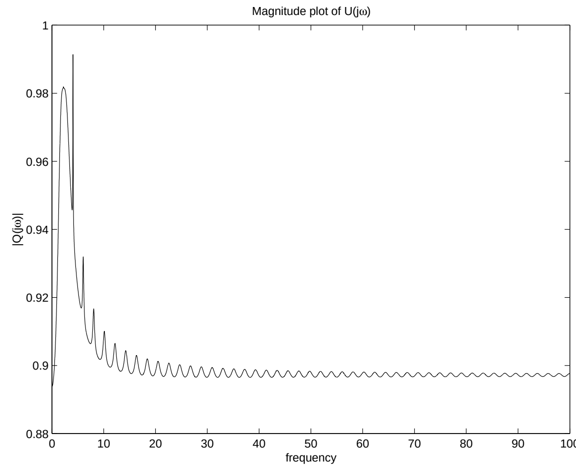

Figure 4: Magnitude plot of $U(j\omega)$

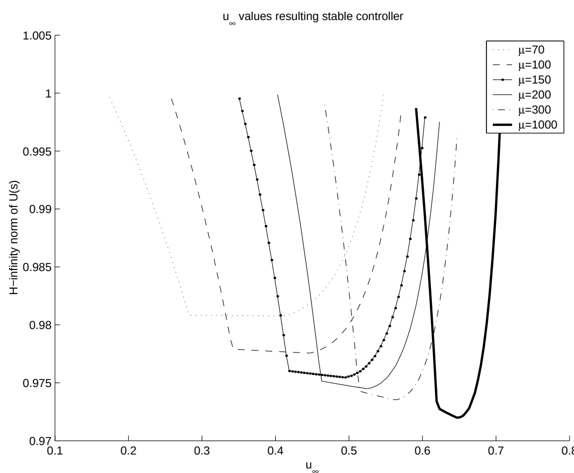

Figure 5: u_∞ values resulting stable $\mathcal{H}^\infty$ controller

By the search algorithm, we can find many u_∞ values for different μ resulting in stable $\mathcal{H}^\infty$ controller at $\rho = 1.94584$ provided that U satisfies the norm condition for chosen $Q = u_\infty$. The various u_∞ values resulting stable $\mathcal{H}^\infty$ controller can be seen in Figure 5. We can observe that as μ is increased, the range of u_∞ stabilizing the controller decreases and the minimum value of $\|U\|_\infty$ in the u_∞ range becomes smaller.

6 Conclusions

In this paper, for delay systems, we investigated stability of the $\mathcal{H}^\infty$ controllers whose stucture is given in [16],[19]. We considered the controllers in two subsections according to their number of poles (finite, infinite). For each case, necessary conditions and design methods based on simple sufficient condition are given to find stable suboptimal $\mathcal{H}^\infty$ controllers.

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