Fixed-order strong H-infinity control of interconnected systems with time-delays

Suat Gumussoy *, **Wim Michiels** *

* *Department of Computer Science, K. U. Leuven, Celestijnenlaan 200A, 3001, Heverlee, Belgium (e-mails: suat.gumussoy@cs.kuleuven.be, wim.michiels@cs.kuleuven.be).*

Abstract: We design fixed-order strong H-infinity controllers for general time-delay systems. The designer chooses the controller order and may introduce constant time-delays in the controller. We represent the closed-loop system of the plant and the controller as delay differential algebraic equations (DDAEs). This representation deals with any interconnection of systems with time-delays without any elimination techniques. We present a numerical algorithm to compute the strong H-infinity norm for DDAEs which is robust to arbitrarily small delay perturbations, unlike the standard H-infinity norm. We optimize the strong H-infinity norm of the closed-loop system based on non-smooth, non-convex optimization methods using this algorithm and the computation of the gradient of the strong H-infinity norm with respect to the controller parameters. We tune the controller parameters and design H-infinity controllers with a prescribed order or structure.

Keywords: fixed-order controller design, strong h-infinity norm, time-delay, interconnected systems, delay differential algebraic equations, computational methods.

1. INTRODUCTION

In control applications, robust controllers are desired to achieve stability and performance requirements under model uncertainties and exogenous disturbances, Zhou et al. (1995). The design requirements are usually defined in terms of $\mathcal{H}_\infty$ norms of the closed-loop functions including the plant, the controller and weights for uncertainties and disturbances. There are robust control methods to design the optimal $\mathcal{H}_\infty$ controller for linear finite dimensional multi-input-multi-output (MIMO) systems based on Riccati equations and linear matrix inequalities (LMIs), see e.g. Doyle et al. (1989); Gahinet and Apkarian (1994) and the references therein. The order of the controller designed by these methods is typically larger or equal than the order of the plant. This is a restrictive condition for high-order plants, since low-order controllers are desired in a practical implementation. The design of fixed-order or low order $\mathcal{H}_\infty$ controller design can be translated into a non-smooth, non-convex optimization problem. Recently fixed-order $\mathcal{H}_\infty$ controllers have been successfully designed for finite dimensional LTI MIMO plants using a direct optimization approach, Gumussoy and Overton (2008). This approach allows the user to choose the controller order and tunes the parameters of the controller to minimize the $\mathcal{H}_\infty$ norm of the objective function. An extension to a class of retarded time-delay systems has been described in Gumussoy and Michiels (2010a).

We design a fixed-order $\mathcal{H}_\infty$ controller in a feedback connection with a time-delay system. The closed-loop system is a delay differential algebraic system and its state-space representation is written as

$$\begin{cases} E\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + Bw(t), \\ z = Cx(t). \end{cases} \quad (1)$$

The time-delays τ_i, $i = 1, \ldots, m$ are positive real numbers and the capital letters are real-valued matrices with appropriate dimensions. The input w and output z are disturbances and signals to be minimized to achieve design requirements and some of system matrices include the controller parameters.

The system with the closed-loop equations (1) represents all interesting cases of the feedback connection of a time-delay plant and a controller. The transformation of the closed-loop system to this form can be easily done by first augmenting the system equations of the plant and controller. As we shall see, this augmented system can subsequently be brought in the form (1) by introducing slack variables to eliminate input/output delays and direct feedthrough terms in the closed-loop equations. Hence, the resulting system of the form (1) is obtained directly without complicated elimination techniques, that may even not be possible in the presence of time-delays.

By interconnecting systems and controller high frequency paths could be created in control loops, which may lead to sensitivity problems with respect to the delays and delay perturbations. Therefore it is important to take the sensitivity explicitly into account in the design. As shown in Gumussoy and Michiels (2010b) the $\mathcal{H}_\infty$ norm of a DAE is not robust against small delay changes. This leads to the definition of the strong $\mathcal{H}_\infty$ norm, the smallest upper bound of the $\mathcal{H}_\infty$ norm that is insensitive to small delay changes, which are inevitable in any practical design due to small modeling errors. Several properties of the strong $\mathcal{H}_\infty$ norm are given in Gumussoy and Michiels (2010b). In this paper, we present a level set algorithm for computing strong $\mathcal{H}_\infty$ norms, see Byers (1988); Boyd and Balakrishnan (1990); Bruinsma and Steinbuch (1990) for the underlying idea behind level set methods. Since time-delay systems are inherently infinite-dimensional systems we adopt a predictor-corrector approach, where the prediction step involves a finite-dimensional approximation of the problem, and the correction step serves to remove the effect of the discretization error on the numerical result.

The numerical algorithm for the norm computation is subsequently applied to the design of fixed-order $\mathcal{H}_\infty$ controllers by a direct optimization approach. In the context of control of LTI systems it is well known that $\mathcal{H}_\infty$ norms are in general non-convex functions of the controller parameters which arise as elements of the closed-loop system matrices. They are typically even not everywhere smooth, although they are differentiable almost everywhere, Gumussoy and Overton (2008). These properties carry over to the case of strong $\mathcal{H}_\infty$ norms of DDAEs under consideration. Therefore, special optimization methods for non-smooth, non-convex problems are required. We will use a combination of BFGS, whose favorable properties in the context of non-smooth problems have been reported in Lewis and Overton (2009), bundle and gradient sampling methods, as implemented in the MATLAB code HANSO [1]. The overall algorithm only requires the evaluation of the objective function, i.e., the strong $\mathcal{H}_\infty$ norm, as well as its derivatives with respect to the controller parameters whenever it is differentiable. The computation of the derivatives is also discussed in the paper.

The presented method is frequency domain based and builds on the eigenvalue based framework developed in Michiels and Niculescu (2007). Time-domain methods for the $\mathcal{H}_\infty$ control of DDAEs have been described in Fridman and Shaked (2002) and the references therein, based on the construction of Lyapunov-Krasovskii functionals.

2. PRELIMINARIES

Let $\text{rank}(E) = n - v$, with $v \le n$, and let the columns of matrix $U \in \mathbb{R}^{n\times v}$, respectively $V \in \mathbb{R}^{n\times v}$, be a (minimal) basis for the left, respectively right nullspace, that is,

$$U^T E = 0, \quad EV = 0. \tag{2}$$

Throughout the paper we make the following assumptions.

Assumption 1. The matrix $U^T A_0 V$ is nonsingular.

Assumption 2. The zero solution of system (1), with $w \equiv 0$, is strongly exponentially stable.

Assumption 1 prevents that the equations (1) are of advanced type and, hence, non-causal. Assumption 2 guarantees that the asymptotic stability of the null solution is robust against small delay perturbations, Hale and Verduyn Lunel (2002).

Finally we use the following notations. Set of nonnegative and strictly positive real numbers are $\mathbb{R}^+, \mathbb{R}_0^+$. The m-tuple time-delays $(\tau_1, \ldots, \tau_m)$ are denoted as $\boldsymbol{\tau}$. The open ball of radius $\epsilon \in \mathbb{R}^+$ centered at $\boldsymbol{\tau} \in (\mathbb{R}^+)^m$ is shown as $\mathcal{B}(\boldsymbol{\tau}, \epsilon) := \{\boldsymbol{\theta} \in (\mathbb{R})^m : \|\boldsymbol{\theta} - \boldsymbol{\tau}\| < \epsilon\}$. The i$^{\text{th}}$ singular value of a matrix A is denoted by $\sigma_i(A)$.

[1] Hybrid Algorithm for non-smooth Optimization, see Overton (2009)

3. MOTIVATING EXAMPLE

We give the following motivating example to illustrate the generality of the system description (1).

Example 3. Consider the feedback interconnection of the system

$$\begin{cases} \dfrac{d}{dt}\left(x(t) + Hx(t-\tau_3)\right) = Ax(t) + B_0 w(t) \\ \qquad\qquad\qquad + B_1 w(t-\tau_1) + B_2 u(t), \\ z(t) = C_z x(t) + D_z w(t), \\ y(t) = C_y x(t) + D_u u(t), \end{cases}$$

and the controller

$$u(t) = Ky(t - \tau_2).$$

The feedback connection has a delay, $\tau_2 \neq 0$, therefore we can not eliminate the output and controller equation. The system is a neutral time-delay system with a direct feedthrough term from w to z, i.e., D_z and an input delay τ_1. By defining a new state X including u and y of the system variables with γ_x and γ_w slack variables, we can easily transform the feedback connection into the form (1).

If we let $X = [\gamma_x^T \; x^T \; u^T \; y^T \; \gamma_w^T]^T$, we can describe the system by the equations

$$\begin{cases} \underbrace{\begin{bmatrix} I & 0_{1\times5} \\ 0_{5\times1} & 0_{5\times5} \end{bmatrix}}_{E} \dot{X}(t) = \underbrace{\begin{bmatrix} 0 & A & B_2 & 0 & 0 & B_0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & C_y & D_y & 0 & -I & 0 \\ -I & I & 0 & 0 & 0 & 0 \\ -I & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}}_{A_0} X(t) \\ \quad + \underbrace{\begin{bmatrix} 0_{1\times5} & B_1 \\ 0_{5\times5} & 0 \end{bmatrix}}_{A_1} X(t-\tau_1) + \underbrace{\begin{bmatrix} 0_{1\times4} & 0 & 0 \\ 0_{1\times4} & K & 0 \\ 0_{4\times4} & 0_{4\times1} & 0_{4\times1} \end{bmatrix}}_{A_2} X(t-\tau_2) \\ \quad + \underbrace{\begin{bmatrix} 0_{3\times1} & 0_{3\times1} & 0_{3\times3} \\ 0 & H & 0_{1\times3} \\ 0_{2\times1} & 0_{2\times1} & 0_{2\times3} \end{bmatrix}}_{A_3} X(t-\tau_3) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}}_{B} w(t) \\ z = \underbrace{\begin{bmatrix} 0 & C_z & 0_{1\times3} & D_z \end{bmatrix}}_{C} X(t) \end{cases}$$

where each element of the above matrices is a block matrix of appropriate dimensions.

Using the technique illustrated with the above example, a broad class of interconnected systems with delays can be brought in the form (1), where the external inputs w and outputs z stem from the performance specifications expressed in terms of appropriately defined transfer functions. The price to pay for the generality of the framework is the increase of the dimension of the system, n, which affects the efficiency of the numerical methods. However, this is a minor problem in most applications because the delay difference equations or algebraic constraints are related to inputs and outputs, and the number of inputs and outputs is usually much smaller than the number of state variables.

4. STRONG $\mathcal{H}_\infty$ NORM AND ITS COMPUTATION

4.1 Definitions and properties

We write the transfer function of the system (1) as

$$T(\lambda) := C\left(\lambda E - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda\tau_i}\right)^{-1} B \qquad (3)$$

and define the *asymptotic* transfer function of the system (1) as

$$T_a(\lambda) := -CV\left(U^T A_0 V + \sum_{i=1}^{m} U^T A_i V e^{-\lambda\tau_i}\right)^{-1} U^T B. \qquad (4)$$

The terminology stems from the fact that the transfer function T and the asymptotic transfer function T_a converge to each other for high frequencies, see Gumussoy and Michiels (2010b).

In Gumussoy and Michiels (2010b), it is shown that the function

$$\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m \mapsto \|T(j\omega, \boldsymbol{\tau})\|_\infty, \qquad (5)$$

is, in general, not continuous, which is inherited from the behavior of the asymptotic transfer function, T_a, more precisely the function

$$\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m \mapsto \|T_a(j\omega, \boldsymbol{\tau})\|_\infty. \qquad (6)$$

Since small modeling errors and uncertainty are inevitable in a practical design, we are interested in the smallest upper bound for the $\mathcal{H}_\infty$ norm which is insensitive to small delay perturbations. A formal definition of the *strong $\mathcal{H}_\infty$ norm* is as follows.

Definition 4. Let $G(\lambda;\boldsymbol{\tau})$ be the transfer function of a strongly stable system. The strong $\mathcal{H}_\infty$ norm of G, $|||G(j\omega,\boldsymbol{\tau})|||_\infty$, is defined as

$$|||\, G(j\omega,\boldsymbol{\tau})|||_\infty := \lim_{\epsilon\to 0+} \sup\{\|G(j\omega,\boldsymbol{\tau}_\epsilon)\|_\infty : \boldsymbol{\tau}_\epsilon \in \mathcal{B}(\boldsymbol{\tau},\epsilon)\cap(\mathbb{R}^+)^m\}.$$

Several properties of the strong $\mathcal{H}_\infty$ norm of T and T_a are listed below.

Proposition 5. The following assertions hold Gumussoy and Michiels (2010b).

(1) For every $\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m$, we have

$$|||T_a(j\omega,\boldsymbol{\tau})|||_\infty = \max_{\boldsymbol{\theta}\in[0,\ 2\pi]^m} \sigma_1\left(\mathbb{T}_a(\boldsymbol{\theta})\right), \qquad (7)$$

where

$$\mathbb{T}_a(\boldsymbol{\theta}) := CV\left(-U^T A_0 V - \sum_{i=1}^{m} U^T A_i V e^{-j\theta_i}\right)^{-1} U^T B. \qquad (8)$$

(2) The function

$$\boldsymbol{\tau} \in (\mathbb{R}_0^+)^m \mapsto |||T(j\omega,\boldsymbol{\tau})|||_\infty \qquad (9)$$

is continuous.

(3) The strong $\mathcal{H}_\infty$ norm of the transfer function T satisfies

$$|||T(j\omega,\boldsymbol{\tau})|||_\infty = \max\left(\|T(j\omega,\boldsymbol{\tau})\|_\infty, |||T_a(j\omega,\boldsymbol{\tau})|||_\infty\right). \qquad (10)$$

(4) Let $\xi > |||T_a(j\omega,\boldsymbol{\tau})|||_\infty$ hold. Then there exist real numbers $\epsilon > 0$, $\Omega > 0$ and an integer N such that for any $\boldsymbol{r} \in \mathcal{B}(\boldsymbol{\tau},\epsilon)\cap(\mathbb{R}^+)^m$, the number of frequencies $\omega^{(i)}$ such that

$$\sigma_k\left(T(j\omega^{(i)},\boldsymbol{r})\right) = \xi, \qquad (11)$$

for some $k \in \{1,\ldots,n\}$, is smaller then N, and, moreover, $|\omega^{(i)}| < \Omega$.

The strong $\mathcal{H}_\infty$ norm of the transfer function T can be computed by (10) depending on the computation of the $\mathcal{H}_\infty$ norm of T and the strong $\mathcal{H}_\infty$ norm of T_a. Therefore, in §4.2, we first give a numerical method for the strong $\mathcal{H}_\infty$ norm computation of the asymptotic transfer function T_a based on the computational formula (7) of Proposition 5. Next, we present the algorithm for computing the strong $\mathcal{H}_\infty$ norm of T in §4.3.

4.2 Strong $\mathcal{H}_\infty$ norm of the asymptotic transfer function

The computation of $|||T_a(j\omega,\boldsymbol{\tau})|||_\infty$ is based on expression (7). We obtain an approximation by restricting $\boldsymbol{\theta}$ in (7) to a grid,

$$|||T_a(j\omega,\boldsymbol{\tau})|||_\infty \approx \max_{\boldsymbol{\theta}\in\Omega_h} \sigma_1\left(\mathbb{T}_a(\boldsymbol{\theta})\right), \qquad (12)$$

where Ω_h is a m-dimensional grids over the hypercube $[0,\ 2\pi]^m$ and $\mathbb{T}_a(\boldsymbol{\theta})$ is defined by (8). If a high accuracy is required, then the approximate results may be corrected to the full precision by solving the nonlinear equations

$$\begin{cases} \begin{bmatrix} \mathbb{A}_{22}(\boldsymbol{\theta}) & -\xi^{-1}U^T BB^T U \\ \xi^{-1}V^T C^T CV & -\mathbb{A}_{22}^*(\boldsymbol{\theta}) \end{bmatrix} \begin{bmatrix} u_a \\ v_a \end{bmatrix} = 0, \\ n(u_a, v_a) = 0, \\ \Re(e^{-j\theta_i}(v_a^* U^T A_i V u_a)) = 0,\ i = 1,\ldots,m, \end{cases} \qquad (13)$$

where

$$\mathbb{A}_{22}(\boldsymbol{\theta}) = -U^T A_0 V - \sum_{i=1}^{m} U^T A_i V e^{-j\theta_i}$$

and $n(u_a, v_a) = 0$ is a normalization constraint. The first equation in (13) implies that ξ is a singular value of $\mathbb{T}_a(\boldsymbol{\theta})$. The last equation of (13) expresses that the derivatives of the singular value ξ with respect to the elements of $\boldsymbol{\theta}$ are zero. In our implementation we solve (13) using the Gauss-Newton method, which exhibits quadratic convergence because the (overdetermined) equations have an exact solution.

In most practical problems, the number of delays to be considered in $\mathbb{A}_{22}(\boldsymbol{\theta})$ is much smaller than the number of system delays, m, because most of the terms are zero. Note that in a control application a nonzero term corresponds to a high frequency feedthrough over the control loop.

4.3 Algorithm

From (10) the following implication can be derived.

$$|||\, T(j\omega,\boldsymbol{\tau})|||_\infty > |||T_a(j\omega,\boldsymbol{\tau})|||_\infty \Rightarrow |||\, T(j\omega,\boldsymbol{\tau})|||_\infty = \|T(j\omega,\boldsymbol{\tau})\|_\infty.$$

Moreover, we know from Statement (4) of Proposition 5 that, given a level

$$\xi > |||T_a(j\omega,\boldsymbol{\tau})|||_\infty, \qquad (14)$$

there are only *finitely* many frequencies ω for which for a singular value of $T(j\omega,\boldsymbol{\tau})$ is equal to ξ. These properties allow a slight adaptation of the level set algorithm for $\mathcal{H}_\infty$ computations of retarded time-delay systems as described in Michiels and Gumussoy (2010), whenever one restricts to the situation where (14) holds. The latter is possible by a preliminary computation of the strong $\mathcal{H}_\infty$ norm of T_a, as outlined in §4.2.

The level set method is based on a predictor-corrector approach. In the prediction (approximation) step the infinite-dimensional problem is discretized allowing to apply methods for LTI systems. In particular, the time-delay system (1) can be approximated by a finite-dimensional system using a spectral method, as in Vanbiervliet et al. (2010). The finite-dimensional system is described as

$$\mathbf{E_N}\dot{z}(t) = \mathbf{A_N}z(t) + \mathbf{B_N}u(t), \ z(t) \in \mathbb{R}^{(N+1)n\times 1} \quad (15)$$

$$y(t) = \mathbf{C_N}z(t) \quad (16)$$

where N is a positive integer for the number of discretization points in the interval $[-\tau_{\max},\ 0]$. The transfer function of (15) is given by

$$T_N(\lambda) := \mathbf{C_N}(\lambda\mathbf{E_N} - \mathbf{A_N})^{-1}\mathbf{B_N}. \quad (17)$$

Further details on the transformation to and the infinite-dimensional system and the discretization into a finite-dimensional system are given in Michiels and Gumussoy (2010).

In the correction step the effect of the approximation on the computed $\mathcal{H}_\infty$ norm is removed. The following algorithm computes the strong $\mathcal{H}_\infty$ norm within the tolerance, tol.

Algorithm

<u>*Prediction step:*</u>

(1) calculate the first level, $\xi_l = |||T_a(j\omega,\boldsymbol{\tau})|||_\infty$,
(2) repeat until break
  (a) set $\xi := \xi_l(1+2\text{tol})$
  (b) compute all $\omega^{(i)} \in \mathbb{R}$ satisfying $\sigma_k\left(T_N(j\omega^{(i)})\right) = \xi$. By (Genin et al., 2002, Proposition 12), this can be done by computing generalized eigenvalues of the pencil

$$\lambda\begin{bmatrix}\mathbf{E_N} & 0\\ 0 & \mathbf{E_N}^T\end{bmatrix} - \begin{bmatrix}\mathbf{A_N} & \xi^{-1}\mathbf{B_N}\mathbf{B_N}^T\\ -\xi^{-1}\mathbf{C_N}^T\mathbf{C_N} & -\mathbf{A_N}^T\end{bmatrix}, \quad (18)$$

  whose imaginary axis eigenvalues are given by $\lambda = j\omega^{(i)}$.
  (c) ***if*** no generalized eigenvalues $j\omega^{(i)}$ of (18) exist, ***then***
    if $\xi_l = |||T_a(j\omega,\boldsymbol{\tau})|||_\infty$, ***then***
      set $|||T(j\omega,\boldsymbol{\tau})|||_\infty = |||T_a(j\omega,\boldsymbol{\tau})|||_\infty$
      quit
    else
      compute $\omega^{(i)} \in \mathbb{R}$ satisfying
      $\sigma_k\left(T_N(j\omega^{(i)})\right) = \xi_l$,
      set $\tilde{\xi} = (\xi+\xi_l)/2$, $\tilde{\omega}^{(i)} = \omega^{(i)}, i = 1,2,\ldots$
      break, go to correction step.
    endif
  else
    calculate $\mu^{(i)} := \sqrt{\omega^{(i)}\omega^{(i+1)}}$, $i = 1,2,\ldots$
    set

$$\xi_l := \max_i \max\left(\sigma_1\left(T_N(j\mu^{(i)})\right), |||T_a(j\omega,\boldsymbol{\tau})|||_\infty\right).$$

  endif

<u>*Correction step:*</u>

(a) Solve the nonlinear equations

$$\begin{cases} H(j\omega,\xi)\begin{bmatrix}u\\ v\end{bmatrix} = 0,\\ n(u,v) = 0,\\ \Im\{v^*(E + \sum_{i=1}^m A_i\tau_i e^{-j\omega\tau_i})u\} = 0,\end{cases} \quad (19)$$

where

$$H(j\omega,\xi) = \begin{bmatrix} j\omega E - A_0 - \sum_{i=1}^m A_i e^{-j\omega\tau_i} & -\xi^{-1}BB^T\\ \xi^{-1}C^TC & (j\omega E + A_0 + \sum_{i=1}^m A_i e^{j\omega\tau_i})^T\end{bmatrix}$$

and $n(u,v) = 0$ is a normalizing condition, with the starting values

$$\omega = \tilde{\omega}^{(i)}, \quad \xi = \tilde{\xi} \text{ and}$$
$$\begin{bmatrix}u\\ v\end{bmatrix} = \arg\min \|H(j\tilde{\omega}^{(i)},\tilde{\xi})v\|/\|v\|;$$

denote the solutions with $(\hat{u}^{(i)},\hat{v}^{(i)},\hat{\omega}^{(i)},\hat{\xi}^{(i)})$, for $i = 1,2,\ldots,$
(b) set $|||T(j\omega)|||_\infty := \max_{1\le i\le p}\hat{\xi}^{(i)}$

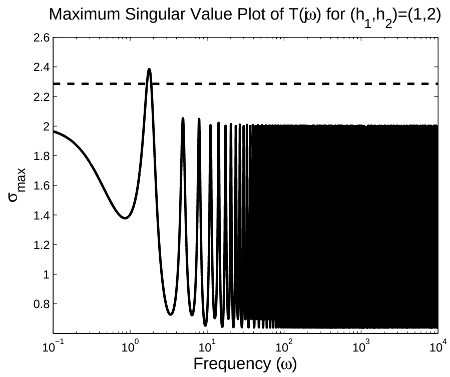

Fig. 1. The maximum singular value plot of $T(j\omega)$ for $(\tau_1,\tau_2) = (1,2)$ as a function of ω.

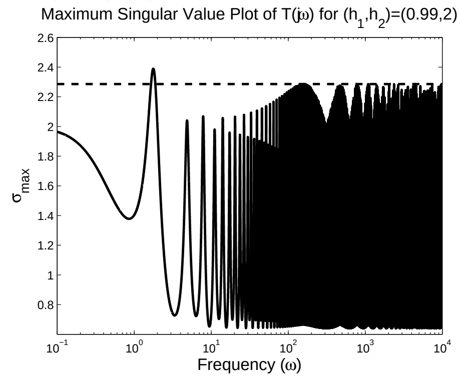

Fig. 2. The maximum singular value plot of $T(j\omega)$ for $(\tau_1,\tau_2) = (0.99,2)$ as a function of ω.

The computational cost of Algorithm in the prediction step is dominated by the computation of the generalized eigenvalues of the pencil matrices with dimensions

$(N+1)2n$. Mathematically, equations (19) characterize extrema in the singular value curves (see Michiels and Gumussoy (2010)), and hence, they can be used to correct peak values. Note that the correction step is only performed if

$$|||T(j\omega,\boldsymbol{\tau})|||_\infty > |||T_a(j\omega,\boldsymbol{\tau})\,|||_\infty\,.$$

In our implementation we solve equations (19) in least squares sense using the Gauss Newton algorithm, which can be shown to be quadratically converging in the case under consideration where the residual in the desired solution is zero.

For details on the choice of the number of discretization points, N, and the tolerance, tol, we refer to Michiels and Gumussoy (2010).

Example 6. Figures 1 and 2 show singular value plots of the transfer function of $(\tau_1,\tau_2)=(1,2)$ and $(\tau_1,\tau_2)=(0.99,2)$ for

$$T(\lambda,\boldsymbol{\tau}) := \frac{\lambda+2}{\lambda(1-1/16e^{-\lambda\tau_1}+1/2e^{-\lambda\tau_2})+1}.$$

The strong $\mathcal{H}_\infty$ norm of the asymptotic transfer function T_a is shown as dashed lines. We use this value as an initial level in the Algorithm. This example also illustrates that the $\mathcal{H}_\infty$ norm of the asymptotic transfer function of a time-delay system may be sensitive to small delay changes as shown in Figures 1 and 2.

5. FIXED-ORDER $\mathcal{H}_\infty$ CONTROLLER DESIGN

The closed-loop system is described as

$$E\dot{x}(t) = A_0(p)x(t) + \sum_{i=1}^{m} A_i(p)x(t-\tau_i) + Bw(t)$$
$$z(t) = Cx(t)$$

where the vector p contains all the parameters in the controller matrices. We design fixed-order $\mathcal{H}_\infty$ controllers by minimizing the strong $\mathcal{H}_\infty$ norm of the closed-loop transfer function T as a function of p. This is a non-convex problem and the objective function (the strong $\mathcal{H}_\infty$ norm) with respect to optimization parameters (controller parameters) is a non-smooth function but its differentiable almost everywhere. Given these properties, we use the non-smooth, non-convex optimization method proposed in Gumussoy and Overton (2008) and implemented as a MATLAB function HANSO in Overton (2009). The optimization algorithm searches for the local minimizer of the objective function in three steps: a quasi-Newton algorithm (in particular, BFGS) to approximate a local minimizer; a local bundle method to verify local optimality for the best point found by BFGS; if this does not succeed, gradient sampling to refine the approximation of the local minimizer, Burke et al. (2006). The optimization algorithm requires the evaluation of the objective function and its gradients with respect to the optimization parameters, whenever it is differentiable. These are described now.

The strong $\mathcal{H}_\infty$ norm of the transfer function T other corresponding parameters are computed by Algorithm 4.3. The derivatives of the norm with respect to controller parameters exist whenever there are unique time-delay values $\hat{\boldsymbol{\theta}}$ or a frequency $\hat{\omega}$ such that

$$|||T(j\omega,\boldsymbol{\tau})|||_\infty = \hat{\xi} = \begin{cases} \sigma_1(\mathbb{T}_a(\hat{\boldsymbol{\theta}})), \text{ if } \hat{\xi} = |||T_a(j\omega,\boldsymbol{\tau})|||_\infty, \\ \sigma_1(\mathbb{T}(j\hat{\omega})), \text{ if } \hat{\xi} > |||T_a(j\omega,\boldsymbol{\tau})|||_\infty \end{cases}$$

holds and, in addition, the largest singular value $\hat{\xi}$ has multiplicity one. We compute the derivative of the strong $\mathcal{H}_\infty$ norm of T with respect to the optimization parameter p_i in the controller matrices as

$$\frac{\partial\xi}{\partial p_i} = \begin{cases} -2\xi^2 \left.\dfrac{\Re\left(v_a^* \frac{\partial \mathbb{A}_{22}(\boldsymbol{\theta})}{\partial p_i} u_a\right)}{v_a^* U^T B B^T U v_a + u_a^* V^T C^T C V u_a}\right|_{(\xi,\boldsymbol{\theta})=(\hat{\xi},\hat{\boldsymbol{\theta}})} \\ \qquad\qquad\qquad \text{if } \hat{\xi} = |||T_a(j\omega,\boldsymbol{\tau})|||_\infty, \\ -2\xi^2 \left.\dfrac{\Re\left(v^* \frac{\partial A(j\omega)}{\partial p_i} u\right)}{v^* B B^T v + u^* C^T C u}\right|_{(\xi,\omega)=(\hat{\xi},\hat{\omega})} \\ \qquad\qquad\qquad \text{if } \hat{\xi} > |||T_a(j\omega,\boldsymbol{\tau})|||_\infty \end{cases}$$

where given $\xi=\hat{\xi}$, u_a, v_a and u, v are vectors in (13) and (19) for $\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}$ and $\omega=\hat{\omega}$ respectively. For detailed derivation on derivative calculations, see Millstone, M. (2006); Gumussoy and Michiels (2010a).

We note that our approach allows constant entries in the controller matrices. Hence, we can impose a structure on the controller, e.g., a PID controller. Although we illustrated our method for a dynamic controller, it can be applied to more general controller structures including time-delays in the controller states or inputs.

6. EXAMPLES

Consider the feedback interconnection of the system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} x(t-h) + \\ &\qquad \begin{pmatrix} -0.5 \\ 1 \end{pmatrix} w(t) + \begin{pmatrix} 3 \\ 1 \end{pmatrix} u(t) \\ z(t) &= \begin{pmatrix} 1 & -0.5 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \\ y(t) &= x(t),\end{aligned}$$

and the controller

$$u(t) = Ky(t).$$

In Fridman and Shaked (1998), a static order controller is designed with $\mathcal{H}_\infty$ performance 0.4436 for the given delays in Table 1. We designed static order controllers using our approach and give their closed-loop $\mathcal{H}_\infty$ performances for various delays in Table 1. We will present our extensive benchmark results in the full version of the paper.

7. CONCLUDING REMARKS

We showed that a very broad class of interconnected systems can be brought in the standard form (1) in a systematic way. Input/output delays and direct feedthrough terms can be dealt with by introducing slack variables. An additional advantage in the context of control design is the linearity of the closed loop matrices w.r.t. the controller parameters.

We presented a predictor-corrector algorithm for the strong $\mathcal{H}_\infty$ norm computation of DDAEs. Based on the numerical algorithm for the strong $\mathcal{H}_\infty$ norm and its gradient computation with respect to controller parameters,

h	ξ	K
0.1	0.4005	$[-17.8065,\ 9.5915]$
0.2	0.3981	$[-7.1854,\ 3.7727]$
0.3	0.3995	$[-4.3068,\ 2.0695]$
0.4	0.4041	$[-3.7321,\ 1.6556]$
0.5	0.4101	$[-3.5878,\ 1.5017]$
0.6	0.4158	$[-3.4104,\ 1.3563]$
0.7	0.4206	$[-3.2772,\ 1.2514]$
0.8	0.3953	$[0.8892,\ -0.9308]$
0.9	0.3953	$[0.0518,\ -0.4074]$
1.0	0.3953	$[0.1942,\ -0.4964]$

Table 1. The achieved $\mathcal{H}_\infty$ performances ξ by static order controllers

we applied non-smooth, non-convex optimization methods for designing controllers with a fixed-order or structure.

ACKNOWLEDGEMENTS

This work has been supported by the Programme of Interuniversity Attraction Poles of the Belgian Federal Science Policy Office (IAP P6- DYSCO), by OPTEC, the Optimization in Engineering Center of the K.U.Leuven, by the project STRT1-09/33 of the K.U.Leuven Research Council and the project G.0712.11N of the Research Foundation - Flanders (FWO).

REFERENCES

Boyd, S. and Balakrishnan, V. (1990). A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its $\mathcal{L}_\infty$-norm. *Systems & Control Letters*, 15, 1–7.

Bruinsma, N. and Steinbuch, M. (1990). A fast algorithm to compute the $\mathcal{H}_\infty$-norm of a transfer function matrix. *Systems and Control Letters*, 14, 287–293.

Burke, J.V., Henrion, D., Lewis, A.S., and Overton, M.L. (2006). Stabilization via nonsmooth, nonconvex optimization. *IEEE Transactions on Automatic Control*, 51(11), 1760–1769.

Byers, R. (1988). A bisection method for measuring the distance of a stable matrix to the unstable matrices. *SIAM Journal on Scientific and Statistical Computing*, 9(9), 875–881.

Doyle, J., Glover, K., P.P., K., and B.A., F. (1989). State-space solutions to standard $\mathcal{H}^2$ and $\mathcal{H}^\infty$ control problems. *IEEE Transactions on Automatic Control*, 34(8), 831–847.

Fridman, E. and Shaked, U. (1998). $\mathcal{H}_\infty$-state-feedback control of linear systems with small state delay. *Systems and Control Letters*, 33, 141–150.

Fridman, E. and Shaked, U. (2002). H_∞-control of linear state-delay descriptor systems: an LMI approach. *Linear Algebra and its Applications*, 351-352, 271–302.

Gahinet, P. and Apkarian, P. (1994). An linear matrix inequality approach to $\mathcal{H}_\infty$ control. *International Journal of Robust and Nonlinear Control*, 4(4), 421–448.

Genin, Y., Stefan, R., and Van Dooren, P. (2002). Real and complex stability radii of polynomial matrices. *Linear Algebra and its Applications*, 351-352, 381–410.

Gumussoy, S. and Michiels, W. (2010a). Fixed-order h-infinity optimization of time-delay systems. In M. Diehl, F. Glineur, E. Jarlebring, and W. Michiels (eds.), *Recent Advances in Optimization and its Applications in Engineering*. Springer.

Gumussoy, S. and Michiels, W. (2010b). On the sensitivity of the h-infinity norm of interconnected systems with time-delays. *submitted to* 18^{th} *World Congress of IFAC 2011. See also Technical Report TW578, Department of Computer Science, K.U.Leuven, 2010.*

Gumussoy, S. and Overton, M. (2008). Fixed-order H-infinity controller design via hifoo, a specialized nonsmooth optimization package. In *Proceedings of the American Control Conference*, 2750–2754. Seattle, USA.

Hale, J. and Verduyn Lunel, S.M. (2002). Strong stabilization of neutral functional differential equations. *IMA J. Math. Control Inf.*, 19, 5–23.

Lewis, A. and Overton, M. (2009). Nonsmooth optimization via BFGS. Available from `http://cs.nyu.edu/overton/papers.html`.

Michiels, W. and Gumussoy, S. (2010). Characterization and computation of h-infinity norms of time-delay systems. *SIAM Journal on Matrix Analysis and Applications*, 31(4), 2093–2115.

Michiels, W. and Niculescu, S.I. (2007). *Stability and stabilization of time-delay systems. An eigenvalue based approach.* SIAM.

Millstone, M. (2006). *HIFOO 1.5: Structured control of linear systems with a non-trivial feedthrough.* Master's thesis, New York University.

Overton, M. (2009). HANSO: a hybrid algorithm for nonsmooth optimization. Available from `http://cs.nyu.edu/overton/software/hanso/`.

Vanbiervliet, J., Michiels, W., and Jarlebring, E. (2010). Using spectral discretization for the optimal h-2 design of time-delay systems. *submitted to International Journal of Control.*

Zhou, K., Doyle, J., and Glover, K. (1995). *Robust and optimal control.* Prentice Hall.