

# Computing $\mathcal{H}_\infty$ Norms of Time-Delay Systems

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**Abstract**—In this paper we consider the computation of  $\mathcal{H}_\infty$  norm of retarded time-delay systems with discrete pointwise state delays. It is well known that in the finite dimensional case  $\mathcal{H}_\infty$  norm of a system is computed using the connection between the singular values of the transfer function and the imaginary axis eigenvalues of an Hamiltonian matrix. We show a similar connection between the singular values of a transfer function of a time-delay system and the imaginary axis eigenvalues of an infinite dimensional operator  $\mathcal{L}_\xi$ . Using spectral methods, this linear operator is approximated with a matrix. The approximate  $\mathcal{H}_\infty$  norm of the time-delay system is calculated using the connection between the imaginary eigenvalues of this matrix and the singular values of a finite dimensional approximation of the time-delay system. Finally the approximate results are corrected by solving a set of equations which are obtained from the reformulation of the eigenvalue problem for  $\mathcal{L}_\xi$  as a finite dimensional nonlinear eigenvalue problem.

## I. INTRODUCTION

In robust control of linear systems, stability and performance criteria are often expressed by  $\mathcal{H}_\infty$  norms of appropriately defined transfer functions. Therefore, the availability of robust methods to compute  $\mathcal{H}_\infty$  norms is essential in a computer aided control system design [11].

The computation of  $\mathcal{H}_\infty$  norm for the finite dimensional plants is based on the relation between the existence of the singular values of the transfer function equal to the fixed value and the existence of the imaginary axis eigenvalues of the corresponding Hamiltonian matrix of the same fixed value [6]. This relation allows the computation of  $\mathcal{H}_\infty$  norm via the well-known level set method [1]. It is possible to set the level for the singular values of the transfer function using the relation above and achieve quadratically convergent algorithms in  $\mathcal{H}_\infty$  norm computation for finite dimensional plants [2], [5].

In this paper, we consider the computation of the  $\mathcal{H}_\infty$  norm of the stable time-delay system  $G$  with the transfer function representation,

$$G(s) = C(sI - A_0 - \sum_{i=1}^m A_i e^{-\tau_i s})^{-1} B + D \quad (1)$$

where the system matrices are  $A_i^{n \times n}$ ,  $B^{n \times n_u}$ ,  $C^{n_y \times n}$ ,  $D^{n_y \times n_u}$ ,  $i = 0, \dots, m$  are real-valued and the time delays,  $(\tau_1, \dots, \tau_m)$ , are nonnegative real numbers. Equivalently, the  $\mathcal{H}_\infty$  norm of (1) is defined as the largest singular value of the  $G(j\omega)$  over all the frequency interval.

In Section II, it is shown that given  $\xi > 0$ , the existence of the singular values of the transfer function (1) equal to  $\xi$  is equivalent to the existence of the imaginary axis eigenvalues of the linear infinite-dimensional operator  $\mathcal{L}_\xi$ .

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By this relation, we extended the level set methods to the time-delay systems. The difference lies in the fact that in every iteration of the level  $\xi$ , the imaginary axis eigenvalues of the infinite-dimensional linear operator  $\mathcal{L}_\xi$  are required instead of that of Hamiltonian matrix in the finite dimensional delay-free case.

In Section III, we approximate the infinite-dimensional operator  $\mathcal{L}_\xi$  by a finite-dimensional matrix approximation  $\mathcal{L}_\xi^N$ . We show that for a fixed level set  $\xi > 0$ , there is a relation between the imaginary axis eigenvalues of the matrix  $\mathcal{L}_\xi^N$  and the singular values of a finite-dimensional approximation of  $G$  equal to  $\xi$ . Therefore, the  $\mathcal{H}_\infty$  norm calculated by the level set methods and  $\mathcal{L}_\xi^N$  is the  $\mathcal{H}_\infty$  norm of the finite dimensional approximation of  $G$ .

In Section IV, we correct the approximate results by using the property that the eigenvalues of the linear infinite dimensional operator  $\mathcal{L}_\xi$  appear as solutions of a finite dimensional nonlinear eigenvalue problem. This allows to write the conditions to characterize the peaks in singular value plot and correct the approximate  $\mathcal{H}_\infty$  norm.

Two numerical algorithms based on level set methods [6], [5] for  $\mathcal{H}_\infty$  norm computation of the time-delay system are given in Section V. A numerical example and concluding remarks are given in Section VI and VII.

### Notation:

The notation in the paper is standard and given below.

- $\mathbb{C}, \mathbb{R}$  : the field of the complex and real numbers,
- $\mathbb{C}^n$  : n-dimensional complex space,
- $A^*$  : complex conjugate transpose of the matrix  $A$ ,
- $A^{-T}$  : transpose of the inverse matrix of  $A$ ,
- $\mathcal{D}(\cdot)$  : domain of an operator,
- $\sigma_i(A)$  :  $i^{\text{th}}$  singular value of  $A$ ,
- $\Re(u)$  : real part of the complex number  $u$ ,
- $\Im(u)$  : imaginary part of the complex number  $u$ .
- $\det(A)$  : determinant of the matrix  $A$ .
- $\tau_{\max}$  : the maximum of the delays  $(\tau_1, \dots, \tau_m)$  in (1).
- $\mathcal{C}$  : the space of continuous complex functions.

## II. LINEAR INFINITE-DIMENSIONAL EIGENVALUE PROBLEM

The connection between the singular values of a transfer function and the imaginary eigenvalues of a corresponding Hamiltonian matrix is given in [6], [2] that laid the basis for the established level set methods to compute  $\mathcal{H}_\infty$  norms. The following theorem generalizes this connection to the time-delay systems:

*Theorem 2.1:* Let  $\xi > 0$  be such that the matrix

$$D_\xi := D^T D - \xi^2 I$$

is non-singular. For  $\omega \geq 0$ , the matrix  $G(j\omega)$  has a singular value equal to  $\xi > 0$  if and only if  $\lambda = j\omega$  is an eigenvalue of the linear infinite dimensional operator  $\mathcal{L}_\xi$  on  $X := \mathcal{C}([-\tau_{\max}, \tau_{\max}], \mathbb{C}^{2n})$  which is defined by

$$\begin{aligned} \mathcal{D}(\mathcal{L}_\xi) &= \{\phi \in X : \phi' \in X, \\ \phi'(0) &= M_0\phi(0) + \sum_{i=1}^m (M_i\phi(-\tau_i) + M_{-i}\phi(\tau_i))\}, \quad (2) \\ \mathcal{L}_\xi\phi &= \phi', \quad \phi \in \mathcal{D}(\mathcal{L}_\xi) \end{aligned} \quad (3)$$

with

$$\begin{aligned} M_0 &= \begin{bmatrix} A_0 - BD_\xi^{-1}D^TC & -BD_\xi^{-1}B^T \\ \xi^2 C^T D_\xi^{-T} C & -A_0^T + C^T D D_\xi^{-1} B^T \end{bmatrix}, \\ M_i &= \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{-i} = \begin{bmatrix} 0 & 0 \\ 0 & -A_i^T \end{bmatrix}, \quad 1 \leq i \leq N. \end{aligned}$$

**Proof.** The proof is given in Appendix, Section IX.

Equivalently Theorem 2.1 can be stated that there is a singular value of  $G$  equal to  $\xi$  at  $\omega = \omega_0$ ,  $\sigma_i(G(j\omega_0)) = \xi$ , if and only if the eigenvalue problem for the linear operator  $\mathcal{L}_\xi$

$$(\lambda I - \mathcal{L}_\xi)u = 0 : \lambda \in \mathbb{C}, u \in X, u \neq 0. \quad (4)$$

has a solution for  $\lambda = j\omega_0$ .

Although the operator  $\mathcal{L}_\xi$  generally has infinite number of eigenvalues, one can show that the number of eigenvalues on the imaginary axis is always finite. Therefore, eigenvalue problem (4) is computationally well-posed.

*Proposition 2.2:*  $\lambda$  is an eigenvalue of the linear operator  $\mathcal{L}_\xi$  if and only if  $-\bar{\lambda}$  is an eigenvalue of the linear operator  $\mathcal{L}_\xi$ .

**Proof.** The proof is given in Appendix, Section IX.

By Proposition 2.2, the set of eigenvalues of  $\mathcal{L}_\xi$  is symmetric with respect to the imaginary axis. In the delay-free case, the operator  $\mathcal{L}_\xi$  reduces to a Hamiltonian matrix.

The key role of Theorem 2.1 is that it reduces the  $\mathcal{H}_\infty$  norm computation of (1) into the bisection search for maximum level set  $\xi$  for which the linear operator  $\mathcal{L}_\xi$  has imaginary axis eigenvalues.

Instead of solving the difficult linear infinite dimensional eigenvalue problem, we can use the connection in Theorem 2.1 and apply the level set methods for the  $\mathcal{H}_\infty$  norm computation of (1) in two steps:

- 1) The approximate solution of the eigenvalue problem can be calculated by solving the standard linear eigenvalue problem of the discretized linear operator of  $\mathcal{L}_\xi$ .
- 2) The approximate results can be corrected by using the property that the eigenvalues of the linear infinite dimensional operator  $\mathcal{L}_\xi$  appear as solutions of a finite dimensional nonlinear eigenvalue problem.

The approximation of the linear operator  $\mathcal{L}_\xi$  and the corresponding standard eigenvalue problem (4) is given in Section III. The correction algorithm of the approximate results in the second step is explained in Section IV.

### III. FINITE-DIMENSIONAL APPROXIMATION

In this section, the linear infinite dimensional eigenvalue problem (4) is discretized based on approximating the infinite-dimensional operator  $\mathcal{L}_\xi$  by a matrix using a *spectral method* (see, e.g. [9], [3], [4]). Given a positive integer  $N$ , we consider a mesh  $\Omega_N$  of  $2N + 1$  distinct points in the interval  $[-\tau_{\max}, \tau_{\max}]$ :

$$\Omega_N = \{\theta_{N,i}, i = -N, \dots, N\}, \quad (5)$$

where

$$-\tau_{\max} \leq \theta_{N,-N} < \dots < \theta_{N,0} = 0 < \dots < \theta_{N,N} \leq \tau_{\max}.$$

This allows to replace the continuous space  $X$  with the space  $X_N$  of discrete functions defined over the mesh  $\Omega_N$ , i.e. any function  $\phi \in X$  is discretized into a block vector  $x = [x_{-N}^T \dots x_N^T]^T \in X_N$  with components

$$x_i = \phi(\theta_{N,i}) \in \mathbb{C}^{2n}, \quad i = -N, \dots, N.$$

Let  $\mathcal{P}_N x$ ,  $x \in X_N$  be the unique  $\mathbb{C}^{2n}$  valued interpolating polynomial of degree  $\leq 2N$  satisfying

$$\mathcal{P}_N x(\theta_{N,i}) = x_i, \quad i = -N, \dots, N.$$

In this way, the operator  $\mathcal{L}_\xi$  over  $X$  can be approximated with the matrix  $\mathcal{L}_\xi^N : X_N \rightarrow X_N$ , defined as

$$\begin{aligned} (\mathcal{L}_\xi^N x)_i &= (\mathcal{P}_N x)'(\theta_{N,i}), \quad i = -N, \dots, -1, 1, \dots, N \\ (\mathcal{L}_\xi^N x)_0 &= M_0 \mathcal{P}_N x(0) + \sum_{i=1}^m (M_i \mathcal{P}_N x(-\tau_i) \\ &\quad + M_{-i} \mathcal{P}_N x(\tau_i)) \end{aligned} \quad (6)$$

Using the Lagrange representation of  $\mathcal{P}_N x$ ,

$$\mathcal{P}_N x = \sum_{k=-N}^N l_{N,k} x_k,$$

where the Lagrange polynomials  $l_{N,k}$  are real valued polynomials of degree  $2N$  satisfying

$$l_{N,k}(\theta_{N,i}) = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases}$$

we obtain the explicit form

$$\mathcal{L}_\xi^N = \begin{bmatrix} d_{-N,-N} & \dots & d_{-N,N} \\ \vdots & & \vdots \\ d_{-1,-N} & \dots & d_{-1,N} \\ a_{-N} & \dots & a_N \\ d_{1,-N} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{N,-N} & \dots & d_{N,N} \end{bmatrix} \in \mathbb{R}^{(2N+1)(2n) \times (2N+1)2n},$$

where

$$\begin{aligned} d_{i,k} &= l'_{N,k}(\theta_{N,i})I, \quad i, k \in \{-N, \dots, N\}, i \neq 0 \\ a_0 &= M_0 x_0 + \sum_{k=1}^m (M_k l_{N,k}(-\tau_k) + M_{-k} l_{N,k}(\tau_k)) \\ a_k &= \sum_{k=1}^m (M_k l_{N,k}(-\tau_k) + M_{-k} l_{N,k}(\tau_k)) \\ k &\in \{-N, \dots, N\}, k \neq 0. \end{aligned}$$

Note that all the problem specific information and the parameter  $\xi$  are concentrated in the middle row of  $\mathcal{L}_\xi^N$ , i.e. the elements  $(a_{-N}, \dots, a_N)$ , while all other elements of  $\mathcal{L}_\xi^N$  can be computed beforehand.

The matrix  $\mathcal{L}_\xi^N$  is a dense matrix with dimensions  $(2N+1)(2n) \times (2N+1)(2n)$ . Using the approach at Section 2.2.2 in [10] based on appropriate choice of the polynomial basis and the grid, the eigenvalue problem for  $\mathcal{L}_\xi^N$  can be written as a sparse generalized eigenvalue problem. Therefore, large-scale methods can be utilized for the linear eigenvalue problem.

Since the methods for computing  $\mathcal{H}_\infty$  norms proposed in [7] are based on checking the presence of eigenvalues of  $\mathcal{L}_\xi^N$  on the imaginary axis and thus strongly rely on the symmetry of the eigenvalues with respect to the imaginary axis, it is important that this property is *preserved* in the discretization. The following Proposition gives the condition on the mesh such that this symmetry holds.

*Proposition 3.1:* If the mesh  $\Omega_N$  satisfies

$$\theta_{N,-i} = -\theta_{N,i}, \quad i = 1, \dots, N, \quad (7)$$

then the following result hold: for all  $\lambda \in \mathbb{C}$ , we have

$$\det(\lambda I - \mathcal{L}_\xi^N) = 0 \Leftrightarrow \det(-\bar{\lambda} - \mathcal{L}_\xi^N) = 0. \quad (8)$$

**Proof.** The proof is given in Appendix, Section IX.

We are primarily interested in the eigenvalues of  $\mathcal{L}_\xi$  on the imaginary axis. These eigenvalues are typically among the smallest eigenvalues and one can easily show that the individual eigenvalues of  $\mathcal{L}_\xi^N$  exhibit spectral convergence to the corresponding eigenvalues of  $\mathcal{L}_\xi$  (following the lines of [3]). Since the symmetry property of the spectrum is preserved in the discretization, a small value of  $N$  is sufficient in most practical problems for computing a good approximation of the  $\mathcal{H}_\infty$ -norm which can be employed as a starting point for a direct computation.

#### IV. CORRECTION OF $\mathcal{H}_\infty$ NORM

By using the finite dimensional level set methods, the largest level set  $\xi$  where  $\mathcal{L}_\xi^N$  has imaginary axis eigenvalues and their corresponding frequencies are computed. In the correction step, these approximate results are corrected by using the property that the eigenvalues of the  $\mathcal{L}_\xi$  appear as solutions of a finite dimensional nonlinear eigenvalue problem. The following theorem establishes the link between the linear infinite dimensional (4) and the nonlinear eigenvalue problem.

*Theorem 4.1:* Let  $\xi > 0$  be such that the matrix

$$D_\xi := D^T D - \xi^2 I$$

is non-singular. Then,  $\lambda$  is an eigenvalue of linear operator  $\mathcal{L}_\xi$  if and only if

$$\det H_\xi(\lambda) = 0, \quad (9)$$

where

$$H_\xi(\lambda) := \lambda I - M_0 - \sum_{i=1}^m (M_i e^{-\lambda \tau_i} + M_{-i} e^{\lambda \tau_i}) \quad (10)$$

and the matrices  $M_0, M_i, M_{-i}$  are defined in Theorem 2.1.

**Proof.** The proof is given in Appendix, Section IX.

By Proposition 2.2 and Theorem 4.1, the eigenvalues of the nonlinear eigenvalue problem (11) are symmetric with respect to the imaginary axis similar to the Hamiltonian matrix in the delay-free case.

The solutions of (9) can be found by solving

$$H_\xi(\lambda) v = 0, \quad \lambda \in \mathbb{C}, \quad v \in \mathbb{C}^{2n}, \quad v \neq 0, \quad (11)$$

which in general has an infinite number of solution.

Theorem 2.1 and 4.1 establish the connections between the singular values of the transfer function of (1), the linear infinite dimensional eigenvalue problem (4), and the nonlinear eigenvalue problem (11).

The correction method is based on the property that if  $\hat{\xi} = \|G(j\omega)\|_{\mathcal{H}_\infty}$ , then (11) has a multiple non-semisimple eigenvalue:

If  $\hat{\xi} \geq 0$  and  $\hat{\omega} \geq 0$  are such that

$$\|G(j\omega)\|_{\mathcal{H}_\infty} = \hat{\xi} = \sigma_1(G(j\hat{\omega})), \quad (12)$$

then setting

$$h_\xi(\lambda) = \det H_\xi(\lambda),$$

the pair  $(\hat{\omega}, \hat{\xi})$  satisfies

$$h_\xi(j\omega) = 0, \quad h'_\xi(j\omega) = 0. \quad (13)$$

These complex-valued equations seem over-determined but this is not the case due to the spectral properties of  $\mathcal{H}_\xi(\lambda)$ . Using the symmetry of the eigenvalues of the nonlinear eigenvalue problem (11) with respect to imaginary axis, we can write the following:

*Corollary 4.2:* For  $\omega \geq 0$ , we have

$$\Im h_\xi(j\omega) = 0 \quad (14)$$

and

$$\Re h'_\xi(j\omega) = 0. \quad (15)$$

**Proof.** From the symmetry property of the eigenvalues with respect to the imaginary axis,

$$h_\xi(\lambda) = h_\xi(-\lambda), \quad h'_\xi(\lambda) = -h'_\xi(-\lambda).$$

Substituting  $\lambda = j\omega$  yields

$$\begin{aligned} h_\xi(j\omega) &= h_\xi(-j\omega) = (h_\xi(j\omega))^*, \\ h'_\xi(j\omega) &= -h'_\xi(-j\omega) = -(h'_\xi(j\omega))^*, \end{aligned}$$

and the assertions follow.  $\square$

Using Corollary 4.2 we can simplify the conditions (13) to:

$$\begin{cases} \Re h_\xi(j\omega) = 0 \\ \Im h'_\xi(j\omega) = 0 \end{cases}. \quad (16)$$

Hence, the pair  $(\hat{\omega}, \hat{\xi})$  satisfying (12) can be directly computed from the two equations (16), e.g. using Newton's method, provided that good starting values are available.

The drawback of working directly with (16) is that an explicit expression for the determinant of  $H_\xi$  is required. To avoid this, let  $u, v \in \mathbb{C}^n$  be such that

$$H_\xi(j\omega) \begin{bmatrix} u \\ v \end{bmatrix} = 0, \quad n(u, v) = 0,$$

where  $n(u, v) = 0$  is a normalizing condition. Given the structure of  $H_\xi$  it can be verified that a corresponding left eigenvector is given by  $[-v^* \ u^*]$ . According to [8], we get

$$h'_\xi(j\omega) = 0 \Leftrightarrow [-v^* \ u^*] H'_\xi(j\omega) \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

A simple computation yields:

$$[-v^* \ u^*] H'_\xi(j\omega) \begin{bmatrix} u \\ v \end{bmatrix} = 2\Im \{ v^* (I + \sum_{i=1}^p A_i \tau_i e^{-j\omega \tau_i}) u \}, \quad (17)$$

which is always real. This is a consequence of the property (15).

Taking into account the above results, we end up with  $4n+3$  real equations

$$\begin{cases} H(j\omega, \xi) \begin{bmatrix} u \\ v \end{bmatrix} = 0, & n(u, v) = 0 \\ \Im \{v^* (I + \sum_{i=1}^p A_i \tau_i e^{-j\omega \tau_i}) u\} = 0 \end{cases} \quad (18)$$

in the  $4n + 2$  unknowns  $\Re(v), \Im(v), \Re(u), \Im(u), \omega$  and  $\xi$ . These equations are still overdetermined because the property (14) is not explicitly exploited in the formulation, unlike the property (15). However, it makes the equations (18) solvable in least squares sense, and the  $(\omega, \xi)$  components have a one-to-one-correspondence with the solutions of (16).

In conclusion, as a result of the approximation step, the largest  $\xi$  for which  $\mathcal{L}_\xi^N$  has the imaginary axis eigenvalues and their corresponding eigenvectors are the approximate results of the largest eigenvalue of  $G$ . Using these results as estimates of  $(\hat{\xi}, \hat{\omega})$  satisfying (12) and  $u$  and  $v$ , we can find the exact values by solving (18). At the end of the correction step, the exact  $\mathcal{H}_\infty$  norm of  $G$  (1) and the achieved frequency are equal to  $\xi = \hat{\xi}$  and  $\omega = \hat{\omega}$  respectively.

## V. ALGORITHM

We present two algorithms which are based on the relations between the singular values of the transfer function  $G(j\omega)$  and the spectrum of the operator  $\mathcal{L}_\xi$ , described in Theorem 2.1 and the correction method based on the nonlinear eigenvalue problem defined in (18). From these relations we get:

$$\|G(j\omega)\|_{\mathcal{H}_\infty} = \sup\{\xi \in \mathbb{R}_+ : \text{operator } \mathcal{L}_\xi \text{ has an eigenvalue on the imaginary axis}\}. \quad (19)$$

The fact that the infinite-dimensional operator  $\mathcal{L}_\xi$  can be approximated with the matrix  $\mathcal{L}_\xi^N$ , as outlined in Section III, and the fact that an estimate of the  $\mathcal{H}_\infty$  norm of  $G$  can be corrected to the true value, as outlined in Section IV, suggest the following predictor-corrector computational scheme:

- 1) for fixed  $N$ , determine

$$\sup\{\xi \in \mathbb{R}_+ : \text{matrix } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\} \quad (20)$$

and determine the corresponding eigenvalues on the imaginary axis;

- 2) correct the results from the previous step by solving the equations (18).

Under a mild condition on the grid, the next theorem allows to interpret step 1 as computing the  $\mathcal{H}_\infty$  norm of an approximation of  $G$ .

*Theorem 5.1:* Assume that the mesh  $\Omega_N$  is symmetric around the zero as given in (7). Let  $p_N$  be the polynomial of the degree  $2N + 1$  satisfying the conditions,

$$\begin{aligned} p_N(0; \lambda) &= 1, \\ p'_N(\theta_i; \lambda) &= \lambda p_N(\theta_i; \lambda), \quad i = -N, \dots, -1, 1, \dots, N. \end{aligned} \quad (21)$$

Let  $\xi > 0$  be such that  $\det(D^T D - \xi^2 I) \neq 0$ . The matrix  $\mathcal{L}_\xi^N$  has an imaginary axis eigenvalue  $\lambda = j\omega$  if and only if  $G_N(j\omega)$  has a singular value equal to  $\xi$  where

$$G_N(j\omega) = C(j\omega I - A_0 - \sum_{i=1}^m A_i p_N(-\tau_i; j\omega))^{-1} B + D.$$

**Proof.** The proof is given in Appendix, Section IX.

**Remark:** As we shall see later, the functions  $p_N(-\tau_i, \lambda)$  are proper rational functions in  $\lambda$ .

In what follows we assume that the grid  $\Omega_N$ , employed in the discretization of  $\mathcal{L}_\xi$ , is symmetric around zero, i.e. it satisfies (7). Theorem 5.1 guarantees that  $\mathcal{L}_\xi^N$  has eigenvalues on the imaginary axis for all

$$\xi \in [\sigma_1(D), \|G_N(j\omega)\|_{\mathcal{H}_\infty}]$$

and no eigenvalues on the imaginary axis for  $\xi > \|G_N(j\omega)\|_{\mathcal{H}_\infty}$ . Thus the supremum in (20) exists.

In the first algorithm, the prediction step is based on the bisection algorithm presented in [6].

*Algorithm 5.2:*

*Input:* system data,  $N$ , symmetric grid  $\Omega_N$ , tolerance  $tol$  for prediction step

*Output:*  $\|G(j\omega)\|_{\mathcal{H}_\infty}$

Prediction step:

- 1) compute a lower bound  $\xi_l$  on  $\|G_N(j\omega)\|_{\mathcal{H}_\infty}$ , e.g.  $\xi_l := \max\{\sigma_1(G(0)), \sigma_1(D), tol\}$   
set upper bound,  $\xi_h := \infty$
- 2) while  $\xi_h - \xi_l > 2 \text{ tol}$ 
  - 2.1 if  $\xi_h = \infty$ , set  $\xi := 2\xi_l$ , else set  $\xi = (\xi_l + \xi_h)/2$
  - 2.2 compute  $\mathcal{E}_\xi$ , the set of eigenvalues of the matrix  $\mathcal{L}_\xi^N$  on the positive imaginary axis
  - 2.3 if  $\mathcal{E}_\xi = \emptyset$ , then  $\xi_h = \xi$ , else  $\xi_l = \xi$

{result: estimate  $(\xi_h + \xi_l)/2$  for  $\|G_N(j\omega)\|_{\mathcal{H}_\infty}$ }

Correction step:

- 3) determine all eigenvalues  $\{j\omega^{(1)}, \dots, j\omega^{(p)}\}$  of  $\mathcal{L}_{\xi_l}^N$  on the positive imaginary axis, and the corresponding eigenvectors  $\{x^{(1)}, \dots, x^{(p)}\}$ .
- 4) for all  $i \in \{1, \dots, p\}$ , solve (18) with starting values  $\begin{bmatrix} u \\ v \end{bmatrix} = x_0^{(i)}$ ,  $\omega = \omega^{(i)}$ ,  $\xi = \xi_l$ ,  $\sigma = 0$   
denote the solution with  $(\tilde{u}^{(i)}, \tilde{v}^{(i)}, \tilde{\omega}^{(i)}, \tilde{\xi}^{(i)})$ .
- 5) set  $\|G(j\omega)\|_{\mathcal{H}_\infty} := \max_{1 \leq i \leq p} \tilde{\xi}^{(i)}$ .

In the prediction step of the second algorithm, we use the fast iterative algorithm given in [5] for the prediction step: Given a transfer function  $G_N$  defined in Theorem 5.1 and its corresponding *Hamiltonian-like* matrix  $\mathcal{L}_\xi^N$  (by Theorem 5.1), the largest singular values of  $G_N$  are calculated as follows:

- For a fixed level set  $\xi$  (shown as dashed lines in Figure 1), calculate the imaginary axis eigenvalues of  $\mathcal{L}_\xi^N$  (shown in gray dots in Figure), these eigenvalues are also the frequencies of the singular values equal to  $\xi$  by Theorem 5.1,
- Find the middle points on each interval of the calculated frequencies (shown with cross signs in Figure), and calculate the largest singular value of each middle point (shown in black dots in the Figure),
- Set the next level set  $\xi$  to the maximum of the calculated largest singular values at the middle points.

This algorithm [5] is quadratically convergent and well known method in the computation of  $\mathcal{H}_\infty$  norms for the finite dimensional systems. The overall algorithm for the computation of  $\mathcal{H}_\infty$  norm of (1) becomes:

*Algorithm 5.3:*

*Input:* system data,  $N$ , symmetric grid  $\Omega_N$ , candidate critical frequency  $\omega_i$  if available,

tolerance  $tol$  for prediction step

*Output:*  $\|G(j\omega)\|_{\mathcal{H}_\infty}$

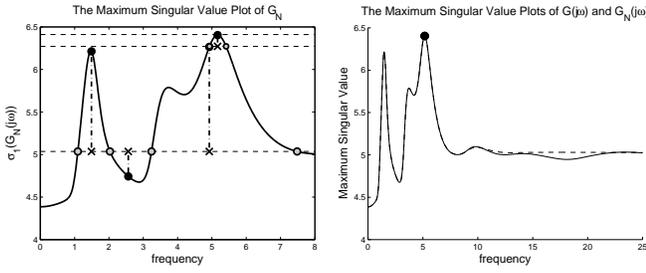


Fig. 1. The maximum singular value plot of the function  $G_N(j\omega)$  Fig. 2. Singular value plot of the function  $G(j\omega)$

**Prediction step:**

- 1) compute a lower bound  $\xi_l$  on  $\|G_N(j\omega)\|_{\mathcal{H}_\infty}$ ,  
e.g.  $\xi_l := \max\{\sigma_1(G(0)), \sigma_1(D), \text{tol}, \sigma_1(G_N(j\omega_i))\}$
  - 2) repeat until break
    - 2.1 set  $\xi := \xi_l(1 + 2 \text{tol})$
    - 2.2 compute the set of eigenvalues  $\mathcal{E}_\xi$  of the matrix  $\mathcal{L}_\xi^N$  on the positive imaginary axis,  
 $\mathcal{E}_\xi := \{j\omega^{(1)}, j\omega^{(2)}, \dots\}$ , with  $0 \leq \omega^{(1)} < \omega^{(2)} < \dots$
    - 2.3 if  $\mathcal{E}_\xi = \emptyset$ , break  
else  
 $\mu^{(i)} := \sqrt{\omega^{(i)}\omega^{(i+1)}}$ ,  $i = 1, 2, \dots$   
compute  $\sigma_1(G_N(j\mu^{(i)}))$ ,  $i = 1, 2, \dots$   
set  $\xi_i := \max_i \sigma_1(G_N(j\mu^{(i)}))$
- {result: estimate  $(\xi + \xi_l)/2$  for  $\|G_N(j\omega)\|_{\mathcal{H}_\infty}$ }

**Correction step:**

follow the steps 3.-5. of Algorithm 5.2

In Step 2.3 of Algorithm 5.3, we need the evaluation of the  $G_N(j\omega)$  at specific frequencies. This can be done as follows:

**Evaluation of  $G_N$**

Algorithm 5.3 relies on the evaluation of the function  $G_N$ , and, hence, on the evaluation of the polynomials  $p_N(-\tau_i; \lambda)$ ,  $i = 1, \dots, m$  for several values of  $\lambda$ . Given the polynomial basis  $B_i(t)$ , we represent  $p_N(\cdot; \lambda)$ :

$$p_N(t; \lambda) = \sum_{i=0}^{2N} \alpha_i B_i(t).$$

From its definition  $p_N(\cdot; \lambda)$  satisfies the conditions

$$p_N(0; \lambda) = 1, \text{ and } p'_N(\theta_i; \lambda) = \lambda p_N(\theta_i; \lambda), \quad i \in \{-N, \dots, N\} \cup \{1, \dots, N\}. \quad (22)$$

For  $\lambda \neq 0$ , the conditions can be written as

$$\left( \lambda \begin{bmatrix} 0_{1 \times (2N+1)} \\ M \end{bmatrix} - \begin{bmatrix} b \\ N \end{bmatrix} \right) \alpha = \begin{bmatrix} -1 \\ 0_{2N \times 1} \end{bmatrix}, \quad (23)$$

where  $M_{ij} = B_{j-1}(\theta_{i-(N-1)})$ ,  $N_{ij} = B'_{j-1}(\theta_{i-(N-1)})$ ,  $b_{1j} = B_{j-1}(0)$  and  $\alpha_{j1} = \alpha_{j-1}$  for  $i = 1, \dots, 2N$  and  $j = 1, \dots, 2N + 1$ .

After solving (23) for a given value of  $\lambda$  we can evaluate

$$p_N(-\tau_i; \lambda) = \sum_{i=0}^{2N} \alpha_i B_i(-\tau_i), \quad i = 0, \dots, m.$$

*Remark 5.4:* Although the prediction step in Algorithm 5.3 corresponds to computing  $\|G_N(\lambda)\|_{\mathcal{H}_\infty}$ , the matrix function  $G_N(\lambda)$  or the rational functions  $p_N(\tau_i; \lambda)$  never need to be explicitly computed (note that they stem from a particular *interpretation* of the effect of a spectral discretization of the operator  $\mathcal{L}_\xi$  into the matrix  $\mathcal{L}_\xi^N$ ). Algorithm 5.3 only relies on computing the eigenvalues of  $\mathcal{L}_\xi^N$  and on evaluating  $G_N(j\omega)$  at *specific frequencies*.

*Remark 5.5:* The definition of  $G_N$  in (22) interprets the term  $p_N(t, \lambda)$  as an approximation of the term  $e^{\lambda t}$  over the whole interval  $[-\tau_{\max}, \tau_{\max}]$  where  $\tau_{\max} = \max\{\tau_1, \dots, \tau_m\}$ . Note that the use of the well-known Padé approximation for the time-delay will cause numerically bad-scaled matrix in  $\mathcal{L}_\xi^N$  due to the different magnitudes in the Padé coefficients. Note that the Padé approximation depends on the time-delay and for multiple delays, each delay is approximated separately which will increase the  $\mathcal{L}_\xi^N$  dimension considerably. However, the term  $p_N(t, \lambda)$  approximates multiple delays with a single term.

*Remark 5.6:* Note that the prediction and correction steps are to some extent independent of each other. In particular, other choices for a finite-dimensional approximation in the prediction step are possible (e.g., using Padé-like approximations or the frequency grid).

*Remark 5.7:* The numerical method for computing  $\mathcal{H}_\infty$  norm can be used for computing  $\mathcal{L}_\infty$  norm of the time-delay system without any modification.

## VI. EXAMPLE

The time-delay system (1) has the dimensions as  $m = 7$ ,  $n = 10$ ,  $n_u = 2$ ,  $n_y = 4$  with delays  $\tau_1 = 0.1$ ,  $\tau_2 = 0.2$ ,  $\tau_3 = 0.3$ ,  $\tau_4 = 0.4$ ,  $\tau_5 = 0.5$ ,  $\tau_6 = 0.6$ ,  $\tau_7 = 0.8$ .

To illustrate insights of the algorithm and results, the maximum singular value plot of the transfer function  $G(j\omega)$  (1) and that of the discretized transfer function  $G_N(j\omega)$  are shown in Figure 2 with blue and red lines where  $N = 6$ . Note that the approximated transfer function has almost same behavior until  $\omega = 10$ . The iterations in the prediction step of the second algorithm can be seen in Figure 1. After three level set iterations, the prediction step yields  $\xi = 6.0436$  and the frequencies  $\omega^{(1)} = 5.1660$  and  $\omega^{(2)} = 5.1666$ . Two frequencies converge to the peak of the maximum singular value plot  $\xi = 6.4040$  at  $\tilde{\omega}^{(1)} = \tilde{\omega}^{(2)} = 5.1662$ . Therefore, the  $\mathcal{H}_\infty$  norm of the time-delay system is  $\|G(j\omega)\|_{\mathcal{H}_\infty} = 6.4040$ .

The problem data for the above benchmark example and a MATLAB implementation of our code for the  $\mathcal{H}_\infty$  norm computation are available at the website <http://www.cs.kuleuven.be/~wimm/software/hinf>.

## VII. CONCLUSION

A numerically stable method to compute  $\mathcal{H}_\infty$  norm of time-delay system with arbitrary number of delays is given. As a generalization of the finite dimensional case, we show the connection between singular values of a transfer function and the eigenvalues of an infinite dimensional linear operator, equivalent to the Hamiltonian matrix in delay free case. By the discretization of the infinite dimensional linear operator, an approximation of  $\mathcal{H}_\infty$  norm of the time-delay system is found. This result is corrected using the equations based on the nonlinear eigenvalue problem. The algorithms are easily extendable to the systems with distributed delays.

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## REFERENCES

- [1] S. Boyd, V. Balakrishnan, and P. Kabamba, "A Bisection Method for Computing the  $\mathcal{H}_\infty$ -Norm of a Transfer Matrix and Related Problems," *Mathematics of Control, Signals, and Systems*, vol.2 (3), pp.207–219, 1989.
- [2] S. Boyd and V. Balakrishnan, "A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its  $\mathcal{L}_\infty$ -norm," *Systems & Control Letters*, vol.15, pp.1–7, 1990.
- [3] D. Breda, S. Maset, and R. Vermiglio, "Pseudospectral differencing methods for characteristic roots of delay differential equations," *SIAM Journal on Scientific Computing*, vol.27 (2), pp.482–495, 2005.
- [4] D. Breda, S. Maset, and R. Vermiglio, "Pseudospectral approximation of eigenvalues of derivative operators with non-local boundary conditions," *Appl. Numer. Math.*, vol.56 (3-4), pp.318–331, 2006.
- [5] N. A. Bruinsma and M. Steinbuch, "A fast algorithm to compute the  $\mathcal{H}_\infty$ -norm of a transfer function matrix," *Systems & Control Letters*, vol.14, pp.287–293, 1990.
- [6] R. Byers, "A bisection method for measuring the distance of a stable matrix to the unstable matrices," *SIAM Journal on Scientific and Statistical Computing*, vol.9 (9), pp.875–881, 1988.
- [7] Y. Genin, R. Stefan, and P. Van Dooren, "Real and complex stability radii of polynomial matrices," *Linear Algebra and its Applications*, vol.351-352, pp.381–410, 2002.
- [8] R. Hryniv and P. Lancaster, "On the perturbation of analytic matrix functions," *Integral Equations and Operator Theory*, vol.34, pp.325–338, 1999.
- [9] L. N. Trefethen, *Spectral methods in MATLAB*, volume 10 of Software, Environments, and Tools, SIAM, 2000.
- [10] K. Verheyden, *Numerical bifurcation analysis of large-scale delay differential equations*, Ph.D. Thesis, K. U. Leuven, March 2007.
- [11] K. Zhou, J.C. Doyle, and K. Glover, *Robust and optimal control*, Prentice Hall, 1995.

## IX. APPENDIX

**Proof of Theorem 4.1.** Assume that  $\mathcal{L}_\xi u = \lambda u$  holds. By (3), we obtain  $u(t) = e^{\lambda t}v$ ,  $t \in [-\tau_{\max}, \tau_{\max}]$ , with  $v \in \mathbb{C}^{2n}$ . Taking into account the boundary condition (2), the nonlinear eigenvalue problem is satisfied,  $H_\xi(\lambda)v = 0$ . Conversely, if  $H_\xi(\lambda)v = 0$ , then it is readily verified that  $u \equiv ve^{\lambda\theta}$ ,  $\theta \in [-\tau_{\max}, \tau_{\max}]$ , belongs to  $\mathcal{D}(\mathcal{L}_\xi)$  and satisfies  $(\mathcal{L}_\xi - \lambda I)u = 0$ .  $\square$

The following theorem shows the connection between the singular value of  $G$  equal to  $\xi$  and the imaginary axis eigenvalue of the nonlinear eigenvalue problem (11).

*Theorem 9.1:* Let  $\xi > 0$  be such that the matrix  $\det(D_\xi) \neq 0$ . For  $\omega \geq 0$ , the matrix  $G(j\omega)$  has a singular value equal to  $\xi$  if and only if  $\lambda = j\omega$  is a solution of the equation

$$\det H_\xi(\lambda) = 0, \quad (24)$$

where  $D_\xi$  and  $H_\xi$  are defined in Theorem 4.1.

**Proof.** The proof is similar to the proof of Proposition 22 in [7]. For all  $\omega \in \mathbb{R}$ , we have the relation

$$\det H_\xi(j\omega) \det D_\xi(j\omega) = \det(G^*(j\omega)G(j\omega) - \xi^2 I) \det \left( \begin{bmatrix} A(j\omega) & 0 \\ 0 & -A(j\omega)^* \end{bmatrix} \right), \quad (25)$$

where  $A(j\omega) = j\omega I - A_0 - \sum_{i=1}^m A_i e^{-j\omega\tau_i}$ . Both left and right hand side can be interpreted as expressions for the determinant of the 2-by-2 block matrix

$$\left[ \begin{array}{cc|c} A(j\omega) & 0 & -B \\ C^T C & -A(j\omega)^* & C^T D \\ \hline D^T C & B^T & D_\xi \end{array} \right]$$

using Schur complements. Since  $D_\xi$  is non-singular and  $G$  is stable, we get from (25):

$$\det(G^*(j\omega)G(j\omega) - \xi^2 I) = 0 \Leftrightarrow \det H_\xi(j\omega) = 0.$$

This is equivalent to the assertion of the theorem.  $\square$

**Proof of Theorem 2.1:** Theorem 2.1 follows from Theorem 9.1 and Theorem 4.1.

**Proof of Proposition 2.2:** It can be verified that

$$H_\xi(-\bar{\lambda}) = - \left( \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes I \right) H_\xi(\lambda) \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I \right) \right)^*,$$

hence,

$$\det H_\xi(-\bar{\lambda}) = (\det H_\xi(\lambda))^*. \quad (26)$$

By Theorem 4.1, the proposition follows.  $\square$

**Proof of Theorem 5.1:** As in the continuous case the discretized linear eigenvalue problem

$$\mathcal{L}_\xi^N x = \lambda x, \quad \lambda \in \mathbb{C}, \quad x \in \mathbb{C}^{(2N+1)2n}, \quad x \neq 0, \quad (27)$$

has a nonlinear eigenvalue problem of dimension  $2n$  as counterpart. To see this, we get from (6) and (27):

$$(\mathcal{P}_N x)'(\theta_{N,i}) = \lambda x_i = \lambda \mathcal{P}_N x(\theta_{N,i})$$

for  $i \in \{-N, \dots, N\}$ ,  $i \neq 0$  (28)

$$M_0 \mathcal{P}_N x(0) + \sum_{i=1}^m (M_i \mathcal{P}_N x(-\tau_i) + M_{-i} \mathcal{P}_N x(\tau_i)) = \lambda x_0 = \lambda \mathcal{P}_N x(0). \quad (29)$$

From  $\mathcal{P}_N x(0) = x_0$  and (28) it follows that

$$\mathcal{P}_N x(\cdot) = p_N(\cdot; \lambda) x_0, \quad (30)$$

where  $p_N(\cdot; \lambda) : \mathbb{R} \rightarrow \mathbb{C}$  is the *collocation polynomial* for the equation

$$\dot{z}(t) = \lambda z(t), \quad z, \lambda \in \mathbb{C}, \quad (31)$$

$v$  which satisfies (31) on  $\Omega_N \setminus \{0\}$ , as well the interpolating condition  $p_N(0; \lambda) = 1$ . Note that for a fixed value of  $t$  the function  $p_N(t; \lambda)$  is a rational function in  $\lambda$ . When substituting (30) in (29) we arrive at the discretized nonlinear eigenvalue problem (32) and (33),

$$H_\xi^N(\lambda) x_0 = 0, \quad (32)$$

where

$$H_\xi^N(\lambda) = \lambda I - M_0 - \sum_{i=1}^m (M_i p_N(-\tau_i; \lambda) + M_{-i} p_N(\tau_i; \lambda)). \quad (33)$$

and the matrices  $M_0, M_i, M_{-i}$  are defined in Theorem 2.1. The nonlinear eigenvalue problem (32) is equivalent to the linear infinite dimensional eigenvalue problem (27). The expressions (32)-(33) can also be interpreted as a direct approximation of (11) and (10).

By Proposition 3.1, the eigenvalues of (27) are symmetric with respect to the imaginary axis. Using the equivalence of (27) and (32), same symmetry property is valid for (32). The assertion follows from the arguments mentioned in the proof of Theorem 9.1.  $\square$

**Proof of Proposition 3.1:** The condition on the mesh assures that

$$p_N(-\tau_i; \lambda) = p_N(\tau_i; -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \forall i = 0, \dots, N. \quad (34)$$

Next, using the same arguments as in the proof of Proposition 2.2 we arrive at

$$\det H_\xi^N(-\bar{\lambda}) = (\det H_\xi^N(\lambda))^*. \quad (35)$$

The Proposition follows from (35) and the arguments mentioned in the proof of Theorem 5.1 on the equivalence of the eigenvalue problems (27) and (32).  $\square$