

# Remarks on Strong Stabilization and Stable $\mathcal{H}^\infty$ Controller Design\*

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**Abstract**—A state space based design method is given to find strongly stabilizing controllers for multi-input-multi-output plants (MIMO). A sufficient condition is derived for the existence of suboptimal stable  $\mathcal{H}^\infty$  controller in terms of linear matrix inequalities (LMI) and the controller order is twice that of the plant. A new parameterization of strongly stabilizing controllers is determined using linear fractional transformations (LFT).

## I. INTRODUCTION

Strong stabilization problem is known as the design of a stable feedback controller which stabilizes the given plant. For practical reasons, a stable controller is desired [1], [2]. In this paper, we derive a simple and effective design method to find stable  $\mathcal{H}^\infty$  controllers for MIMO systems.

A stable controller can be designed if and only if the plant satisfies the parity interlacing property (PIP) [3] i.e., the plant has even number of poles between any pair of its zeros on the extended positive real axis. There are several design procedure for strongly stabilizing controllers, [4–16].

The result in this paper is the generalization of the work in [11] using LMIs. The procedure is quite simple, efficient and easy to solve by using the LMI Toolbox of MATLAB [17]. In the next section, it is shown that if a certain LMI has a feasible solution, then it is possible to obtain a stable  $\mathcal{H}^\infty$  controller whose order is twice the order of the plant. Moreover, a parameterization of strongly stabilizing controllers can be given in terms of LFT.

The paper is organized as follows. The main results are given in Section 2. Stable  $\mathcal{H}^\infty$  controller design procedure is proposed in Section 3. Numerical examples and comparison with other methods can be found in Section 4 and concluding remarks are made in the last section.

### Notation

The notation is fairly standard. A state space realization of a transfer function,  $G(s) = C(sI - A)^{-1}B + D$ , is shown by  $G(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  and the linear fractional transformation of  $G$  by  $K$  is denoted by  $\mathcal{F}_l(G, K)$  which is equivalent to  $G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$  where  $G$  is partitioned as  $G = \left[ \begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array} \right]$ . As a shorthand notation for LMI

expressions, we will define  $\Gamma(A, B) := B^T A^T + AB$  where  $A, B$  are matrices with compatible dimensions.

## II. STRONG STABILIZATION OF MIMO SYSTEMS

Consider the standard feedback system with generalized plant,  $G$ , which has state space realization,

$$G(s) = \left[ \begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_{11} & D_{12} \\ C & D_{21} & 0 \end{array} \right]. \quad (\text{II.1})$$

where  $A \in \mathcal{R}^{n \times n}$ ,  $D_{12} \in \mathcal{R}^{p_1 \times m_2}$ ,  $D_{21} \in \mathcal{R}^{p_2 \times m_1}$  and other matrices are compatible with each other. We suppose the plant satisfies the standard assumptions,

- A.1  $(A, B)$  is stabilizable and  $(C, A)$  is detectable,
- A.2  $\begin{bmatrix} A - \lambda I & B \\ C_1 & D_{12} \end{bmatrix}$  has full column rank for all  $\text{Re}\{\lambda\} \geq 0$ ,
- A.3  $\begin{bmatrix} A - \lambda I & B_1 \\ C & D_{21} \end{bmatrix}$  has full row rank for all  $\text{Re}\{\lambda\} \geq 0$ ,
- A.4  $A$  has no eigenvalues on the imaginary axis.

Let the controller has state space realization,  $K_G(s) = \left[ \begin{array}{c|c} A_K & B_K \\ \hline C_K & 0 \end{array} \right]$  where  $A_K \in \mathcal{R}^{n \times n}$ ,  $B_K \in \mathcal{R}^{n \times p_2}$  and  $C_K \in \mathcal{R}^{m_2 \times n}$ . Define the matrix  $X \in \mathcal{R}^{n \times n}$ ,  $X = X^T \geq 0$  as the stabilizing solution of

$$A^T X + X A - X B B^T X = 0 \quad (\text{II.2})$$

(i.e.,  $A - B B^T X$  is stable) and the “A-matrix” of the closed loop system as  $A_{CL} = \left[ \begin{array}{cc} A & B C_K \\ B_K C & A_K \end{array} \right]$ . Note that since  $(A, B)$  is stabilizable,  $X$  is unique and  $A_X := (A - B B^T X)$  is stable. Also, the closed loop stability is equivalent to whether  $A_{CL}$  is stable or not.

*Lemma 2.1:* Assume that the plant (II.1) satisfies the assumptions A.1 – A.4. There exists a stable stabilizing controller,  $K_G \in \mathcal{RH}^\infty$  if there exists  $X_K \in \mathcal{R}^{n \times n}$ ,  $X_K = X_K^T > 0$  and  $Z \in \mathcal{R}^{n \times p_2}$  for some  $\gamma_K > 0$  satisfying the LMIs,

$$\Gamma(X_K, A) + \Gamma(Z, C) < 0, \quad (\text{II.3})$$

$$\left[ \begin{array}{ccc} \Gamma(X_K, A_X) + \Gamma(Z, C) & -Z & -X B \\ -Z^T & -\gamma_K I & 0 \\ -B^T X & 0 & -\gamma_K I \end{array} \right] < 0, \quad (\text{II.4})$$

where  $X$  is the stabilizing solution of (II.2) and  $A_X$  is as defined previously. Moreover, a stable controller has state

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space realization,  $K_G(s) = \left[ \begin{array}{c|c} \frac{A_X + X_K^{-1}ZC}{-B^T X} & \frac{-X_K^{-1}Z}{0} \end{array} \right]$  and this controller satisfies  $\|K_G\|_\infty < \gamma_K$ .

*Proof:* By using similarity transformation, one can show that  $A_{CL}$  is stable if and only if  $A_X$  and  $A_Z := A + X_K^{-1}ZC$  is stable. Since  $X$  is a stabilizing solution,  $A_X$  is stable. If we rewrite the LMI (II.3) as

$$(A + X_K^{-1}ZC)^T X_K + X_K(A + X_K^{-1}ZC) < 0,$$

it can be seen that  $A_Z$  is stable since  $X_K > 0$ . The second LMI (II.4) comes from KYP lemma and guarantees that  $\|K_G\|_\infty < \gamma_K$ . ■

**Remark 1** If the design only requires the stability of closed loop system, it is enough to satisfy the LMI (II.3), (1, 1) block of (II.4), i.e.,

$$A_X^T X_K + X_K A_X + C^T Z^T + ZC < 0 \quad (\text{II.5})$$

and the controller has same structure as above.

**Remark 2** The Lemma (2.1) is generalization of Theorem 2.1 in [11]. If the algebraic riccati equation (ARE) (7) in [11] has a stabilizing solution,  $Y = Y^T \geq 0$ , then there exists a stable controller in the form,  $\left[ \begin{array}{c|c} \frac{A_X - \gamma_K^2 Y C^T C}{-B^T X} & \frac{\gamma_K^2 Y C^T}{0} \end{array} \right]$ . This structure is the special case of the LMIs (II.3) and (II.4) when  $X_K = (\gamma_K Y)^{-1}$  and  $Z = -\gamma_K C^T$ . Note that our formulation does not assume special structure on  $Z$ . Also in [11], the stability of  $A_Z$  is guaranteed by the same riccati equation, we satisfy the stability condition of  $A_Z$  with another LMI (II.3) which is less restrictive. Therefore, the Lemma (2.1) is less conservative as will be demonstrated in examples.

*Corollary 2.1:* Assume that the sufficient condition (II.3) and (II.5) holds. Then all controllers in the set

$$K_{G,ss} := \{K = \mathcal{F}_l(K_{G,ss}^0, Q) : Q \in \mathcal{RH}^\infty, \|Q\|_\infty < \gamma_Q\}$$

are strongly stabilizing where

$$K_{G,ss}^0(s) = \left[ \begin{array}{c|cc} \frac{A_X + X_K^{-1}ZC}{-B^T X} & \frac{-X_K^{-1}Z}{0} & \frac{B}{I} \end{array} \right]. \quad (\text{II.6})$$

and  $\gamma_Q = (\|C(sI - (A_X + X_K^{-1}ZC))^{-1}B\|_\infty)^{-1}$ .

*Proof:* The result is direct consequence of parameterization of all stabilizing controllers [19]. ■

### III. STABLE $\mathcal{H}^\infty$ CONTROLLER DESIGN FOR MIMO SYSTEMS

The standard  $\mathcal{H}^\infty$  problem is to find a stabilizing controller  $K$  such that  $\|\mathcal{F}_l(P, K)\|_\infty \leq \gamma$  where  $\gamma > 0$  is the closed loop performance level and  $P$  is the generalized plant. It is well known that if two AREs have unique positive semidefinite solutions and the spectral radius condition is satisfied, then standard  $\mathcal{H}^\infty$  problem is solvable.

All suboptimal  $\mathcal{H}^\infty$  controllers can be parameterized as  $K = \mathcal{F}_l(M_\infty, Q)$  where the central controller is in the form

$$M_\infty(s) = \left[ \begin{array}{c|cc} \frac{A_c}{C_{c1}} & \frac{B_{c1}}{D_{c11}} & \frac{B_{c2}}{D_{c12}} \\ \hline & \frac{D_{c21}}{D_{c21}} & 0 \end{array} \right]$$

and  $Q$  is free parameter satisfying  $Q \in \mathcal{RH}^\infty$  and  $\|Q\|_\infty \leq \gamma$ . For derivation and calculation of  $M_\infty$ , see [18], [19].

If we consider  $M_\infty$  as plant and  $\gamma = \gamma_K$ , by using Lemma (2.1), we can find a strictly proper stable  $K_{M_\infty}$  stabilizing  $M_\infty$  and resulting stable  $\mathcal{H}^\infty$  controller,  $C_\gamma = \mathcal{F}_l(M_\infty, K_{M_\infty})$  where  $\|K_{M_\infty}\|_\infty < \gamma_K$ . If sufficient conditions (II.3) and (II.4) are satisfied, then  $K_{M_\infty}$  can be written as,

$$K_{M_\infty}(s) = \left[ \begin{array}{c|c} \frac{A_c - B_{c2}B_{c2}^T X_c + X_K^{-1}Z_c C_{c2}}{-B_{c2}^T X_c} & \frac{-X_K^{-1}Z_c}{0} \end{array} \right]$$

and by similarity transformation, we can obtain the state space realization of  $C_\gamma$  as,

$$C_\gamma(s) = \left[ \begin{array}{c|c} \frac{A_{C_\gamma}}{C_{C_\gamma}} & \frac{B_{C_\gamma}}{D_{c11}} \end{array} \right]$$

where  $X_c$  is the stabilizing solution of

$$A_c^T X_c + X_c A_c - X_c B_{c2} B_{c2}^T X_c = 0$$

as in (II.2) and  $X_{K_c}$ ,  $Z_c$  are the solution of (II.3), (II.4) respectively and the matrices,

$$\begin{aligned} A_{C_\gamma} &= \begin{bmatrix} A_c - B_{c2}B_{c2}^T X_c & -B_{c2}B_{c2}^T X_c \\ 0 & A_c + X_K^{-1}Z_c C_{c2} \end{bmatrix} \\ B_{C_\gamma} &= \begin{bmatrix} B_{c1} \\ -B_{c1} - X_K^{-1}Z_c D_{c21} \end{bmatrix} \\ C_{C_\gamma} &= \begin{bmatrix} C_{c1} - D_{c12}B_{c2}^T X_c & -D_{c12}B_{c2}^T X_c \end{bmatrix} \end{aligned}$$

Note that  $C_\gamma$  is stable stabilizing controller such that  $\|\mathcal{F}_l(P, C_\gamma)\|_\infty < \gamma$ .

## IV. NUMERICAL EXAMPLES AND COMPARISONS

### A. Strong stabilization

The numerical example is chosen from [11]. In order to see the performance of our method, we calculated the minimum  $\gamma_K$  satisfying the sufficient conditions in Lemma (2.1) for the following plants:

$$\begin{aligned} G_1(s) &= \left[ \begin{array}{c} \frac{(s+5)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-\alpha)(s-20)} \\ \frac{(s+1)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-\alpha)(s-20)} \\ \frac{(s+1)(s-2-j\alpha)(s-2+j\alpha)}{(s+2+j)(s+2-j)(s-1)(s-5)} \\ \frac{(s+5)(s-2-j\alpha)(s-2+j\alpha)}{(s+2+j)(s+2-j)(s-1)(s-5)} \end{array} \right] \\ G_2(s) &= \left[ \begin{array}{c} \frac{(s+5)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-\alpha)(s-20)} \\ \frac{(s+1)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-\alpha)(s-20)} \\ \frac{(s+1)(s-2-j\alpha)(s-2+j\alpha)}{(s+2+j)(s+2-j)(s-1)(s-5)} \\ \frac{(s+5)(s-2-j\alpha)(s-2+j\alpha)}{(s+2+j)(s+2-j)(s-1)(s-5)} \end{array} \right] \end{aligned}$$

For various  $\alpha$  values, the minimum  $\gamma_K$  is found. Figure 1 and 2 illustrates the conservatism of [11] mentioned in Remark 2 (where  $\rho_{min}$  is the minimum value of the free parameter  $\gamma_K$  corresponding to the method of [11]).

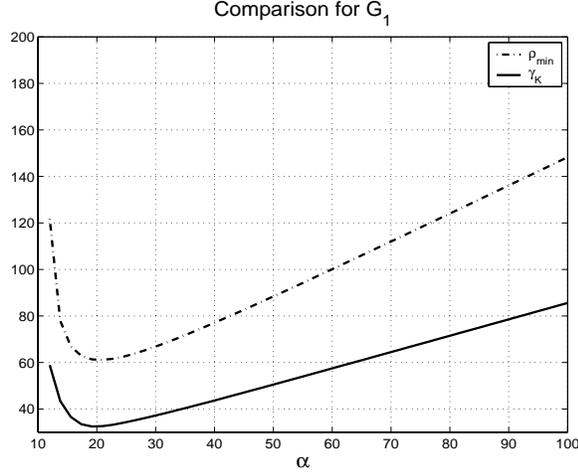


Fig. 1. Comparison for plant  $G_1$

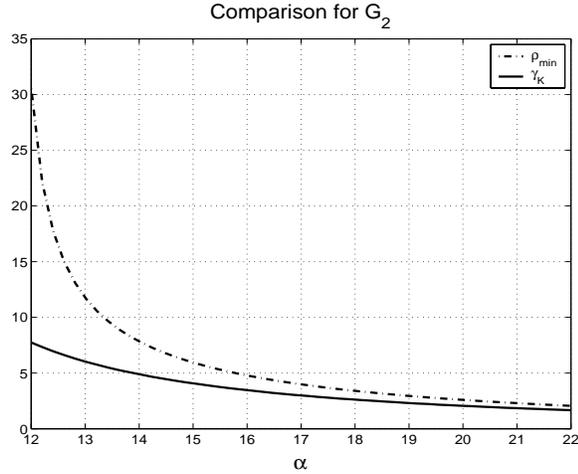


Fig. 2. Comparison for plant  $G_2$

### B. Stable $\mathcal{H}^\infty$ controllers

We applied our method to stable  $\mathcal{H}^\infty$  controller design. As a common benchmark example, the following system is taken from [15]:

$$P = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] \quad (\text{IV.7})$$

where

$$A = \begin{bmatrix} -2 & 1.7321 \\ 1.7321 & 0 \end{bmatrix}$$

$$\left[ \begin{array}{c|c} B_1 & B_2 \end{array} \right] = \begin{bmatrix} 0.1 & -0.1 & 1 \\ -0.5 & 0.5 & 0 \end{bmatrix}$$

$$\left[ \begin{array}{c} C_1 \\ C_2 \end{array} \right] = \begin{bmatrix} 0.2 & -1 \\ 0 & 0 \\ \hline 10 & 11.5470 \end{bmatrix}$$

$$\left[ \begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & 0 \end{array} \right] = \left[ \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0.7071 & 0.7071 & 0 \end{array} \right]$$

The optimal  $\gamma$  value for standard  $\mathcal{H}^\infty$  problem is  $\gamma_{opt} = 1.2929$ . Using the synthesis in [15], a stable  $\mathcal{H}^\infty$  controller is found at  $\gamma_{min} = 1.36994$ . When our method applied, we reached stable  $\mathcal{H}^\infty$  controller for  $\gamma_{K,min} = 1.36957$ . Although it seems slight improvement, our method is much more simpler with help of LMI problem formulation. Apart from standard problem solution (finding  $M_\infty$ ), the algorithm in [15] finds the stable  $\mathcal{H}^\infty$  controller by solving an additional  $\mathcal{H}^\infty$  problem.

Another common benchmark example is from suppression of thermoacoustical instabilities in a combustion chamber and this example is used to find stable  $\mathcal{H}^\infty$  controller in [10], [12]. The generalized plant is described by the equation shown at the top of the next page.

In [10], it is noted that for this problem, the sufficient condition in [7] is not satisfied for even large values of  $\gamma$  and the method is not applicable. As we can see from Table I, the performance of our method is better than the method in [10] except the last case. For all cases, the result of [12] is superior from all other methods. However, the controller order in [12] is 24 which is greater than our controller order, 16. This brings extra freedom in [12] and gives better results. Furthermore, design of the method in [12] involves solution of an additional  $\mathcal{H}^\infty$  problem which is complicated compared to our simple LMI formulation. Also if the algorithm in [12] fails, selection of a new parameter  $Q$  is suggested which is an ad-hoc procedure. Although the performance of the controller suggested in the paper is slightly worse, it is numerically stable and easily formulated.

The following example is taken from [13]. Stable  $\mathcal{H}^\infty$  controller is design for the real plant,

$$P(s) = \frac{s^2 + 0.1s + 0.1}{(s - 0.1)(s - 1)(s^2 + 2s + 3)}.$$

For given plant, assume that the state space representation is as  $P = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ . For sensitivity and complementary sensitivity function minimization problem, the generalized plant can be written as shown in [13]. The comparison of the methods [10], [13], [14] and our method can be seen in Table II. There is a compromise between the methods. The performance of the method in [13] is worse than our

TABLE I  
STABLE  $\mathcal{H}^\infty$  CONTROLLER DESIGN FOR COMBUSTION CHAMBER

$\beta$	Gumussoy-Ozbay(GO)			
	$\gamma_{opt}$	$\gamma_{GO}$	$\gamma_{ZO}$	$\gamma_{CZ}$
0.1	0.232	0.241	0.245	0.237
0.01	0.142	0.176	0.178	0.151
0.001	0.122	0.170	0.170	0.132

$$\begin{aligned}
z_1 &= \frac{0.03s^7 + 0.008s^6 + 0.19s^5 + 0.037s^4 + 0.36s^3 + 0.05s^2 + 0.18s + 0.015}{s^8 + 0.161s^7 + 6s^6 + 0.582s^5 + 9.984s^4 + 0.407s^3 + 3.9822s^2}(w_1 + u), \\
z_2 &= \beta u, \\
z_2 &= \frac{0.0064s^5 + 0.0024s^4 + 0.071s^3 + s^2 + 0.1045s + 1}{s^8 + 0.161s^7 + 6s^6 + 0.582s^5 + 9.984s^4 + 0.407s^3 + 3.9822s^2}(w_1 + u).
\end{aligned}$$

method, but the order of the our controller has twice order of the controller in [13]. Although the method in [14] gives better results than our method, the order of the controller in [14] is considerably higher than our controller order. However, this example clearly shows that our method is superior than [10].

As a remark, the method also gives very good results for single-input-single-output systems. The SISO case example is taken from [11]. For the given plant and weight functions,

$$\begin{aligned}
P(s) &= \frac{(s+5)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-20)(s-30)}, \\
W_1(s) &= \frac{1}{s+1}, \\
W_2(s) &= 0.2,
\end{aligned}$$

the optimal  $\mathcal{H}^\infty$  problem is defined as

$$\gamma_{opt} = \inf_{K \text{ stabilizing } P} \left\| \begin{bmatrix} W_1(1+PK)^{-1} \\ W_2K(1+PK)^{-1} \end{bmatrix} \right\|_\infty$$

and the optimal performance for given data is  $\gamma_{opt} = 34.24$ . The stable  $\mathcal{H}^\infty$  controller can be found for  $\gamma = 42.51$  with the method in [11], whereas our method, which can be seen as a generalization of [11], gives  $\gamma = 35.29$ . In numerical simulations, we observed that when  $\gamma$  approaches to the minimum value satisfying sufficient conditions, the solutions of algebraic riccati equations of [11] become numerically ill-posed. However, the LMI based solution proposed here does not have such a problem.

## V. CONCLUDING REMARKS

In this paper, sufficient conditions for strong stabilization of MIMO systems are obtained and applied to stable  $\mathcal{H}^\infty$  controller design. Our conditions are based on linear matrix inequalities which can be easily solved by the LMI Toolbox of MATLAB. The method is very efficient from numerical point of view as demonstrated with examples. The benchmark examples show that the proposed method is a significant improvement over the existing techniques available in the literature. The exceptions to this claim are

the methods of [12], [14]. In [12], the controller design is based on ad-hoc search method and both methods, [13], [14], result in higher order controllers than the one designed by our method.

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TABLE II  
STABLE  $\mathcal{H}^\infty$  CONTROLLER DESIGN FOR EXAMPLE IN [13]

	Gumussoy-Ozbay	[10]	[13]	[14], Thm 7
$\gamma_{min}$	32.557	37.551	43.167	21.787
Order	$2n$	$2n$	$n$	$3n$