Fixed-Order H-infinity Optimization of Time-Delay Systems

Suat Gumussoy[1] and Wim Michiels[2]

[1] Department of Computer Science, K. U. Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium, suat.gumussoy@cs.kuleuven.be

[2] Department of Computer Science, K. U. Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium, wim.michiels@cs.kuleuven.be

Summary. H-infinity controllers are frequently used in control theory due to their robust performance and stabilization. Classical H-infinity controller synthesis methods for finite dimensional LTI MIMO plants result in high-order controllers for high-order plants whereas low-order controllers are desired in practice. We design fixed-order H-infinity controllers for a class of time-delay systems based on a non-smooth, non-convex optimization method and a recently developed numerical method for H-infinity norm computations.

Robust control techniques are effective to achieve stability and performance requirements under model uncertainties and exogenous disturbances [16]. In robust control of linear systems, stability and performance criteria are often expressed by H-infinity norms of appropriately defined closed-loop functions including the plant, the controller and weights for uncertainties and disturbances. The optimal H-infinity controller minimizing the H-infinity norm of the closed-loop functions for finite dimensional multi-input-multi-output (MIMO) systems is computed by Riccati and linear matrix inequality (LMI) based methods [8, 9]. The order of the resulting controller is equal to the order of the plant and this is a restrictive condition for high-order plants. In practical implementations, fixed-order controllers are desired since they are cheap and easy to implement in hardware and non-restrictive in sampling rate and bandwidth. The fixed-order optimal H-infinity controller synthesis problem leads to a non-convex optimization problem. For certain closed-loop functions, this problem is converted to an interpolation problem and the interpolation function is computed based on continuation methods [1]. Recently fixed-order H-infinity controllers are successfully designed for finite dimensional LTI MIMO plants using a non-smooth, non-convex optimization method [10]. This approach allows the user to choose the controller order and tunes the parameters of the controller to minimize the H-infinity norm of the objective function using the norm value and its derivatives with respect to the controller parameters. In our work, we design fixed-order H-infinity controllers for a class of time-delay systems based on a non-smooth, non-convex optimization method and a recently developed H-infinity norm computation method [13].

1 Problem Formulation

We consider time-delay plant G determined by equations of the form,

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + B_1w(t) + B_2u(t-\tau_{m+1}) \quad (1)$$

$$z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t) \quad (2)$$

$$y(t) = C_2x(t) + D_{21}w(t) + D_{22}u(t-\tau_{m+2}). \quad (3)$$

where all system matrices are real with compatible dimensions and $A_0 \in \mathbb{R}^{n\times n}$. The input signals are the exogenous disturbances w and the control signals u. The output signals are the controlled signals z and the measured signals y. All system matrices are real and the time-delays are positive real numbers. In robust control design, many design objectives can be expressed in terms of norms of closed-loop transfer functions between appropriately chosen signals w to z.

The controller K has a fixed-structure and its order n_K is chosen by the user *apriori* depending on design requirements,

$$\dot{x}_K(t) = A_Kx_K(t) + B_Ky(t) \quad (4)$$

$$u(t) = C_Kx_K(t) \quad (5)$$

where all controller matrices are real with compatible dimensions and $A_K \in \mathbb{R}^{n_K\times n_K}$.

By connecting the plant G and the controller K, the equations of the closed-loop system from w to z are written as,

$$\begin{aligned} \dot{x}_{cl}(t) &= A_{cl,0}x_{cl}(t) + \sum_{i=1}^{m+2} A_{cl,i}x_{cl}(t-\tau_i) + B_{cl}w(t) \\ z(t) &= C_{cl}x_{cl}(t) + D_{cl}w(t) \end{aligned} \quad (6)$$

where

$$\begin{aligned} A_{cl,0} &= \begin{pmatrix} A_0 & 0 \\ B_KC_2 & A_K \end{pmatrix}, \ A_{cl,i} = \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix} \text{ for i} = 1,\dots,\text{m}, \\ A_{cl,m+1} &= \begin{pmatrix} 0 & B_2C_K \\ 0 & 0 \end{pmatrix}, A_{cl,m+2} = \begin{pmatrix} 0 & 0 \\ 0 & B_KD_{22}C_K \end{pmatrix}, \\ B_{cl} &= \begin{pmatrix} B_1 \\ B_KD_{21} \end{pmatrix}, \ C_{cl} = \begin{pmatrix} C_1 & D_{12}C_K \end{pmatrix}, \ D_{cl} = D_{11}. \end{aligned} \quad (7)$$

The closed-loop matrices contain the controller matrices (A_K, B_K, C_K) and these matrices can be tuned to achieve desired closed-loop characteristics.

The transfer function from w to z is,

$$T_{zw}(s) = C_{cl}\left(sI - A_{cl,0} - \sum_{i=1}^{m+2} A_{cl,i}e^{-\tau_i s}\right)^{-1} B_{cl} + D_{cl} \quad (8)$$

and we define fixed-order H-infinity optimization problem as the following.

Problem Given a controller order n_K, find the controller matrices (A_K, B_K, C_K) stabilizing the system and minimizing the H-infinity norm of the transfer function T_{zw}.

2 Optimization Problem

2.1 Algorithm

The optimization algorithm consists of two steps:

1. **Stabilization:** minimizing the spectral abscissa, the maximum real part of the characteristic roots of the closed-loop system. The optimization process can be stopped when the controller parameters are found that stabilizes T_{zw} and these parameters are the feasible points for the H-infinity optimization of T_{zw}.
2. **H-infinity optimization:** minimizing the H-infinity norm of T_{zw} using the starting points from the stabilization step.

If the first step is successful, then a feasible point for the H-infinity optimization is found, i.e., a point where the closed-loop system is stable. If in the second step the H-infinity norm is reduced in a quasi-continuous way, then the feasible set cannot be left under mild controllability/observability conditions.

Both objective functions, the spectral abscissa and the H-infinity norm, are non-convex and not everywhere differentiable but smooth almost everywhere [15]. Therefore we choose a hybrid optimization method to solve a non-smooth and non-convex optimization problem, which has been successfully applied to design fixed-order controllers for the finite dimensional MIMO systems [10].

The optimization algorithm searches for the local minimizer of the objective function in three steps [5]:

1. A quasi-Newton algorithm (in particular, BFGS) provides a fast way to approximate a local minimizer [12],
2. A local bundle method attempts to verify local optimality for the best point found by BFGS,
3. If this does not succeed, gradient sampling [6] attempts to refine the approximation of the local minimizer, returning a rough optimality measure.

The non-smooth, non-convex optimization method requires the evaluation of the objective function, in the second step this is the H-infinity norm of T_{zw} and the gradient of the objective function with respect to controller parameters where it exists. Recently a predictor-corrector algorithm has been developed to compute the H-infinity norm of time-delay systems [13]. We computed the gradients using the derivatives of singular values at frequencies where the H-infinity norm is achieved. Based on the evaluation of the objective function and its gradients, we apply the optimization method to compute fixed-order controllers. The computation of H-infinity norm of time-delay systems (8) is discussed in the following section.

2.2 Computation of the H-infinity Norm

We implemented a predictor-corrector type method to evaluate H-infinity norm of T_{zw} in two steps (for details we refer to [13]):

- **Prediction step:** we calculate the approximate H-infinity norm and corresponding frequencies where the highest peak values in the singular value plot occur.
- **Correction step:** we correct the approximate results from the predicted step.

Theoretical Foundation

The following theorem generalizes the well-known relation between the existence of the singular values of the transfer function equal to the fixed value and the existence of the imaginary axis eigenvalues of a corresponding Hamiltonian matrix [7] to the time-delay systems:

Theorem 1. *[13] Let $\xi > 0$ be such that the matrix*

$$D_\xi := D_{cl}^T D_{cl} - \xi^2 I$$

is non-singular and define $\tau_{\max}$ as the maximum of the delays $(\tau_1, \ldots, \tau_{m+2})$. For $\omega \geq 0$, the matrix $T_{zw}(j\omega)$ has a singular value equal to $\xi > 0$ if and only if $\lambda = j\omega$ is an eigenvalue of the linear infinite dimensional operator $\mathcal{L}_\xi$ on $X := \mathcal{C}([-\tau_{\max},\ \tau_{\max}], \mathbb{C}^{2n})$ which is defined by

$$\mathcal{D}(\mathcal{L}_\xi) = \left\{\phi \in X : \phi' \in X,\ \phi'(0) = M_0\phi(0) + \sum_{i=1}^{m+2}(M_i\phi(-\tau_i) + M_{-i}\phi(\tau_i))\right\}, \quad (9)$$

$$\mathcal{L}_\xi \phi = \phi',\ \phi \in \mathcal{D}(\mathcal{L}_\xi) \quad (10)$$

with

$$M_0 = \begin{bmatrix} A_{cl,0} - B_{cl}D_\xi^{-1}D_{cl}^T C_{cl} & -B_{cl}D_\xi^{-1}B_{cl}^T \\ \xi^2 C_{cl}^T D_\xi^{-T} C_{cl} & -A_{cl,0}^T + C_{cl}^T D_{cl} D_\xi^{-1} B_{cl}^T \end{bmatrix},$$

$$M_i = \begin{bmatrix} A_{cl,i} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{-i} = \begin{bmatrix} 0 & 0 \\ 0 & -A_{cl,i}^T \end{bmatrix}, \quad 1 \leq i \leq m+2.$$

By Theorem 1, the computation of H-infinity norm of T_{zw} can be formulated as an eigenvalue problem for the linear operator $\mathcal{L}_\xi$.

Corollary 1.
$\|T_{zw}\|_\infty = \sup\{\xi > 0 :$ *operator $\mathcal{L}_\xi$ has an eigenvalue on the imaginary axis*$\}$

Conceptually Theorem 1 allows the computation of H-infinity norm via the well-known level set method [2, 4]. However, $\mathcal{L}_\xi$ is an infinite dimensional operator. Therefore, we compute the H-infinity norm of the transfer function T_{zw} in two steps:

1) The prediction step is based on a matrix approximation of $\mathcal{L}_\xi$.

2) The correction step is based on reformulation of the eigenvalue problem of $\mathcal{L}_\xi$ as a nonlinear eigenvalue problem of a finite dimension.

The approximation of the linear operator $\mathcal{L}_\xi$ and the corresponding standard eigenvalue problem for Corollary 1 is given in Section 2.3. The correction algorithm of the approximate results in the second step is explained in Section 2.4.

2.3 Prediction Step

The infinite dimensional operator $\mathcal{L}_\xi$ is approximated by a matrix $\mathcal{L}_\xi^N$. Based on the numerical methods for finite dimensional systems [2, 4], the H-infinity norm of the transfer function T_{zw} can be computed approximately as

Corollary 2.
$\|T_{zw}\|_\infty \approx \sup\{\xi > 0 :$ *operator* $\mathcal{L}_\xi^N$ *has an eigenvalue on the imaginary axis*$\}$.

The infinite-dimensional operator $\mathcal{L}_\xi$ is approximated by a matrix using a *spectral method* (see, e.g. [3]). Given a positive integer N, we consider a mesh Ω_N of $2N+1$ distinct points in the interval $[-\tau_{\max},\ \tau_{\max}]$:

$$\Omega_N = \{\theta_{N,i},\ i = -N, \ldots, N\}, \tag{11}$$

where

$$-\tau_{\max} \le \theta_{N,-N} < \ldots < \theta_{N,0} = 0 < \cdots < \theta_{N,N} \le \tau_{\max}.$$

This allows to replace the continuous space X with the space X_N of discrete functions defined over the mesh Ω_N, i.e. any function $\phi \in X$ is discretized into a block vector $x = [x_{-N}^T \cdots\ x_N^T]^T \in X_N$ with components

$$x_i = \phi(\theta_{N,i}) \in \mathbb{C}^{2n}, \quad i = -N, \ldots, N.$$

Let $\mathcal{P}_N x,\ x \in X_N$ be the unique $\mathbb{C}^{2n}$ valued interpolating polynomial of degree $\le 2N$ satisfying

$$\mathcal{P}_N x(\theta_{N,i}) = x_i, \quad i = -N, \ldots, N.$$

In this way, the operator $\mathcal{L}_\xi$ over X can be approximated with the matrix $\mathcal{L}_\xi^N :\ X_N \to X_N$, defined as

$$\begin{aligned}
\left(\mathcal{L}_\xi^N\ x\right)_i &= (\mathcal{P}_N x)'\,(\theta_{N,i}), \quad i = -N, \ldots, -1, 1, \ldots, N, \\
\left(\mathcal{L}_\xi^N\ x\right)_0 &= M_0 \mathcal{P}_N x(0) + \sum_{i=1}^{m+2} (M_i \mathcal{P}_N x(-\tau_i) + M_{-i} \mathcal{P}_N x(\tau_i)).
\end{aligned}$$

Using the Lagrange representation of $\mathcal{P}_N x$,

$$\mathcal{P}_N x = \textstyle\sum_{k=-N}^{N} l_{N,k}\ x_k,$$

where the Lagrange polynomials $l_{N,k}$ are real valued polynomials of degree $2N$ satisfying

$$l_{N,k}(\theta_{N,i}) = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases}$$

we obtain the explicit form

$$\mathcal{L}_\xi^N = \begin{bmatrix} d_{-N,-N} & \dots & d_{-N,N} \\ \vdots & & \vdots \\ d_{-1,-N} & \dots & d_{-1,N} \\ a_{-N} & \dots & a_N \\ d_{1,-N} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{N,-N} & \dots & d_{N,N} \end{bmatrix} \in \mathbb{R}^{(2N+1)(2n)\times(2N+1)2n},$$

where

$$\begin{aligned} d_{i,k} &= l'_{N,k}(\theta_{N,i})I, \quad i,k \in \{-N,\dots,N\},\ i \neq 0 \\ a_0 &= M_0\ x_0 + \textstyle\sum_{k=1}^{m+2} \left(M_k l_{N,0}(-\tau_k) + M_{-k} l_{N,0}(\tau_k)\right), \\ a_i &= \textstyle\sum_{k=1}^{m+2} \left(M_k l_{N,i}(-\tau_k) + M_{-k} l_{N,i}(\tau_k)\right),\ k \in \{-N,\dots,N\},\ k \neq 0. \end{aligned}$$

2.4 Correction Step

By using the finite dimensional level set methods, the largest level set ξ where $\mathcal{L}_\xi^N$ has imaginary axis eigenvalues and their corresponding frequencies are computed. In the correction step, these approximate results are corrected by using the property that the eigenvalues of the $\mathcal{L}_\xi$ appear as solutions of a finite dimensional nonlinear eigenvalue problem. The following theorem establishes the link between the linear infinite dimensional eigenvalue problem for $\mathcal{L}_\xi$ and the nonlinear eigenvalue problem.

Theorem 2. *[13] Let $\xi > 0$ be such that the matrix*

$$D_\xi := D_{cl}^T D_{cl} - \xi^2 I$$

is non-singular. Then, λ is an eigenvalue of linear operator $\mathcal{L}_\xi$ if and only if

$$\det H_\xi(\lambda) = 0, \tag{12}$$

where

$$H_\xi(\lambda) := \lambda I - M_0 - \sum_{i=1}^{m+2} \left(M_i e^{-\lambda\tau_i} + M_{-i} e^{\lambda\tau_i}\right) \tag{13}$$

and the matrices M_0, M_i, M_{-i} are defined in Theorem 1.

The correction method is based on the property that if $\hat{\xi} = \|T_{zw}(j\omega)\|_\infty$, then (13) has a multiple non-semisimple eigenvalue. If $\hat{\xi} \geq 0$ and $\hat{\omega} \geq 0$ are such that

$$\|T_{zw}(j\omega)\|_{\mathcal{H}_\infty} = \hat{\xi} = \sigma_1(T_{zw}(j\hat{\omega})), \tag{14}$$

then setting

$$h_\xi(\lambda) = \det H_\xi(\lambda),$$

the pair $(\hat{\omega}, \hat{\xi})$ satisfies

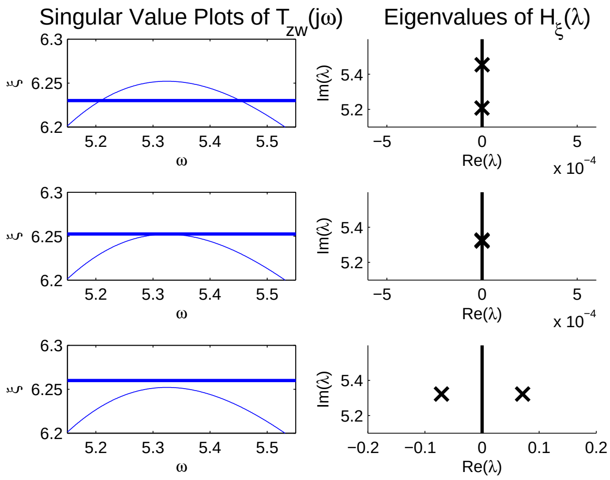

Fig. 1. (left) Intersections of the singular value plot of T_{zw} with the horizontal line $\xi = c$, for $c < \hat{\xi}$ (top), $c = \hat{\xi}$ (middle) and $c > \hat{\xi}$ (bottom). (right) Corresponding eigenvalues of $H_{\xi}(\lambda)$ (13).

$$h_{\xi}(j\omega) = 0, \quad h'_{\xi}(j\omega) = 0. \tag{15}$$

This property is clarified in Figure 1.

The drawback of working directly with (15) is that an explicit expression for the determinant of H_{ξ} is required. This scalar-valued conditions can be equivalently expressed in a matrix-based formulation.

$$\begin{cases} H(j\omega,\ \xi) \begin{bmatrix} u, \\ v \end{bmatrix} = 0, \quad n(u,v) = 0, \\ \Im\left\{v^* \left(I + \sum_{i=1}^{m+1} A_{cl,i}\tau_i e^{-j\omega\tau_i}\right) u\right\} = 0 \end{cases} \tag{16}$$

where $n(u,v) = 0$ is a normalizing condition. The approximate H-infinity norm and its corresponding frequencies can be corrected by solving (16). For further details, see [13].

2.5 Computing the Gradients

The optimization algorithm requires the derivatives of H-infinity norm of the transfer function T_{zw} with respect to the controller matrices whenever it is differentiable. Define the H-infinity norm of the function T_{zw} as

$$f(A_{cl,0}, \ldots, A_{cl,m+2}, B_{cl}, C_{cl}, D_{cl}) = \|T_{zw}(j\omega)\|_{\infty}.$$

These derivatives exist whenever there is a unique frequency $\hat{\omega}$ such that (14) holds, and, in addition, the largest singular value $\hat{\xi}$ of $T_{zw}(j\hat{\omega})$ has multiplicity one. Let w_l and w_r be the corresponding left and right singular vector, i.e.

$$\begin{aligned} T_{zw}(j\hat{\omega})\ w_r &= \hat{\xi}\ w_l, \\ w_l^*\ T_{zw}(j\hat{\omega}) &= \hat{\xi}\ w_r^*. \end{aligned} \tag{17}$$

When defining $\frac{\partial f}{\partial A_{cl,0}}$ as a n-by-n matrix whose (k,l)-th element is the derivative of f with respect to the (k,l)-th element of $A_{cl,0}$, and defining the other derivatives in a similar way, the following expressions are obtained [14]:

$$\begin{aligned} \frac{\partial f}{\partial A_{cl,0}} &= \frac{\Re\left(M(j\hat{\omega})^* C_{cl}^T w_l w_r^* B_{cl}^T M(j\hat{\omega})^*\right)}{w_r^* w_r}, \\ \frac{\partial f}{\partial A_{cl,i}} &= \frac{\Re\left(M(j\hat{\omega})^* C_{cl}^T w_l w_r^* B_{cl}^T M(j\hat{\omega})^* e^{j\omega\tau_i}\right)}{w_r^* w_r} \text{ for } i = 1, \dots, m+2, \\ \frac{\partial f}{\partial B_{cl}} &= \frac{\Re(M(j\hat{\omega})^* C_{cl}^T w_l w_r^*)}{w_r^* w_r}, \frac{\partial f}{\partial C_{cl}} = \frac{\Re(w_l w_r^* B_{cl}^T M(j\hat{\omega})^*)}{w_r^* w_r}, \\ \frac{\partial f}{\partial D_{cl}} &= \frac{\Re\left(w_l w_r^*\right)}{w_r^* w_r} \end{aligned}$$

where $M(j\omega) = \left(j\omega I - A_{cl,0} - \sum_{i=1}^{m+2} A_{cl,i} e^{-j\omega\tau_i}\right)^{-1}$.

We compute the gradients with respect to the controller matrices as

$$\begin{aligned} \frac{\partial f}{\partial A_K} &= \left[\, 0_{n_K \times n}\ I_{n_K} \right] \frac{\partial f}{\partial A_{cl,0}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix}, \\ \frac{\partial f}{\partial B_K} &= \left[\, 0_{n_K \times n}\ I_{n_K} \right] \frac{\partial f}{\partial A_{cl,0}} \begin{bmatrix} I_n \\ 0_{n_K \times n} \end{bmatrix} C_2^T \\ &\quad + \left[\, 0_{n_K \times n}\ I_{n_K} \right] \frac{\partial f}{\partial A_{cl,m+2}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix} C_K^T D_{22}^T + \left[\, 0_{n_K \times n}\ I_{n_K} \right] \frac{\partial f}{\partial B_{cl}} D_{21}^T, \\ \frac{\partial f}{\partial C_K} &= B_2^T \left[\, I_n\ 0_{n \times n_K} \right] \frac{\partial f}{\partial A_{cl,m+1}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix} \\ &\quad + D_{22}^T B_K^T \left[\, 0_{n_K \times n}\ I_{n_K} \right] \frac{\partial f}{\partial A_{cl,m+2}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix} + D_{12}^T \frac{\partial f}{\partial C_{cl}} \begin{bmatrix} 0_{n \times n_K} \\ I_{n_K} \end{bmatrix} \end{aligned}$$

where the matrices I_n, I_{n_K} and $0_{n \times n_K}$, $0_{n_K \times n}$ are identity and zero matrices.

3 Examples

We consider the time-delay system with the following state-space representation,

$$\begin{aligned} \dot{x}(t) &= -x(t) - 0.5x(t-1) + w(t) + u(t), \\ z(t) &= x(t) + u(t), \\ y(t) &= x(t) + w(t). \end{aligned}$$

We designed the first-order controller, $n_K = 1$,

$$\begin{aligned} \dot{x}_K(t) &= 3.61 x_K(t) + 1.39 y(t), \\ u(t) &= -0.83 x_K(t) \end{aligned}$$

achieving the closed-loop H-infinity norm 0.064. The closed-loop H-infinity norms of fixed-order controllers for $n_K = 2$ and $n_K = 3$ are 0.021 and 0.020 respectively.

Our second example is a 4^{th}-order time-delay system. The system contains 4 delays and has the following state-space representation,

$$\dot{x}(t) = \begin{pmatrix} -4.4656 & -0.4271 & 0.4427 & -0.1854 \\ -0.8601 & -5.6257 & 0.8577 & -0.5210 \\ 0.9001 & -0.7177 & -6.5358 & 0.0417 \\ -0.6836 & 0.0242 & 0.4997 & -3.5618 \end{pmatrix} x(t) + \begin{pmatrix} 0.6848 & -0.0618 & 0.5399 & 0.5057 \\ 0.3259 & -0.3810 & 0.6592 & -0.0066 \\ 0.6325 & 0.3752 & 0.4122 & 0.7303 \\ 0.5878 & 0.9737 & 0.1907 & -0.8639 \end{pmatrix} x(t-3.2)$$
$$+ \begin{pmatrix} 0.9371 & -0.7859 & 0.1332 & 0.7429 \\ -0.8025 & 0.4483 & 0.6226 & 0.0152 \\ 0.0940 & 0.2274 & 0.1536 & 0.5776 \\ -0.1941 & 0.5659 & 0.8881 & -0.0539 \end{pmatrix} x(t-3.4) + \begin{pmatrix} 0.6576 & -0.8543 & -0.3460 & 0.6415 \\ -0.3550 & 0.5024 & 0.6081 & 0.9038 \\ 0.9523 & 0.6624 & 0.0765 & -0.8475 \\ -0.4436 & 0.8447 & -0.0734 & 0.4173 \end{pmatrix} x(t-3.9)$$
$$+ \begin{pmatrix} 1 & 0 \\ -1.6 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} w(t) + \begin{pmatrix} 0.2 \\ -1 \\ 0.1 \\ -0.4 \end{pmatrix} u(t-0.2)$$

$$z(t) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0.1 & 1 \\ -1 & 0.2 \end{pmatrix} w(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} -2 & 0.1 \end{pmatrix} w(t) + 0.4u(t-0.2)$$

When $n_K = 1$, our method finds the controller achieving the closed-loop H-infinity norm 1.2606,

$$\dot{x}_K(t) = -0.712 x_K(t) - 0.1639 y(t),$$
$$u(t) = -0.2858 x_K(t)$$

and the results for $n_K = 2$ and $n_K = 3$ are 1.2573 and 1.2505 respectively.

4 Concluding Remarks

We successfully designed fixed-order H-infinity controllers for a class of time-delay systems. The method is based on non-smooth, non-convex optimization techniques and allows the user to choose the controller order as desired. Our approach can be extended to general time-delay systems. Although we illustrated our method for a dynamic controller, it can be applied to more general controller structures. The only requirement is that the closed-loop matrices should depend smoothly on the controller parameters. On the contrary, the existing controller design methods optimizing the closed-loop H-infinity norm are based on Lyapunov theory and linear matrix inequalities, which are conservative if the form of the Lyapunov functions are restricted and requires full state information.

5 Acknowledgements

This article presents results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Ministers Office for Science, Technology and Culture, the Optimization in Engineering Centre OPTEC of the K.U.Leuven, and the project STRT1-09/33 of the K.U.Leuven Research Foundation.

References

1. A. Blomqvist, A. Lindquist and R. Nagamune (2003) Matrix-valued Nevanlinna-Pick interpolation with complexity constraint: An optimization approach. IEEE Transactions on Automatic Control, 48:2172–2190.
2. S. Boyd and V. Balakrishnan (1990) A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its $\mathcal{L}_\infty$-norm. Systems & Control Letters, 15:1–7.
3. D. Breda, S. Maset and R. Vermiglio (2006) Pseudospectral approximation of eigenvalues of derivative operators with non-local boundary conditions. Applied Numerical Mathematics, 56:318–331.
4. N.A. Bruinsma and M. Steinbuch (1990) A fast algorithm to compute the $\mathcal{H}_\infty$-norm of a transfer function matrix. Systems & Control Letters, 14:287–293.
5. J.V. Burke, D. Henrion, A.S. Lewis and M.L. Overton (2006). Stabilization via nonsmooth, nonconvex optimization. IEEE Transactions on Automatic Control, 51:1760-1769.
6. J.V. Burke, A.S. Lewis and M.L. Overton, (2003) A robust gradient sampling algorithm for nonsmooth, nonconvex optimization. SIAM Journal on Optimization, 15:751–779.
7. R. Byers (1988) A bisection method for measuring the distance of a stable matrix to the unstable matrices. SIAM Journal on Scientific and Statistical Computing, 9:875–881.
8. J.C. Doyle, K. Glover, P.P. Khargonekar and B.A. Francis (1989) State-Space solutions to standard $\mathcal{H}^2$ and $\mathcal{H}^\infty$ control problems. IEEE Transactions on Automatic Control 46:1968–1972.
9. P. Gahinet and P. Apkarian (1994) An Linear Matrix Inequality Approach to $\mathcal{H}_\infty$ Control. International Journal of Robust and Nonlinear Control 4:421–448.
10. S. Gumussoy and M.L. Overton, (2008) Fixed-Order H-Infinity Controller Design via HIFOO, a Specialized Nonsmooth Optimization Package. Proceedings of the American Control Conference 2750-2754.
11. R. Hryniv and P. Lancaster (1999) On the perturbation of analytic matrix functions. Integral Equations and Operator Theory, 34:325–338.
12. A.S. Lewis and M.L.Overton, (2009) Nonsmooth optimization via BFGS. Submitted to SIAM Journal on Optimization.
13. W. Michiels and S. Gumussoy, (2009) Computation of H-infinity Norms for Time-Delay Systems. Accepted to SIAM Journal on Matrix Analysis and Applications. See also Technical Report TW551, Department of Computer Science, K.U.Leuven, 2009.
14. M. Millstone, (2006) HIFOO 1.5: Structured control of linear systems with a non-trivial feedthrough. Master's Thesis, New York University.
15. J. Vanbiervliet, K. Verheyden, W. Michiels and S. Vandewalle (2008) A nonsmooth optimization approach for the stabilization of time-delay systems. ESAIM Control, Optimisation and Calcalus of Variations 14:478–493.
16. K. Zhou, J.C. Doyle and K. Glover (1995) Robust and optimal control. Prentice Hall.