# Computing H-infinity Norm of Time-Delay Systems


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## 1 Abstract

We consider the computation of the $\mathscr{H}_\infty$ norm of the stable transfer function $G$,

$$G(j\omega) = C\left(j\omega I - A_0 - \sum_{i=1}^{m} A_i e^{-j\omega\tau_i}\right)^{-1} B + De^{-j\omega\tau_0} \quad (1)$$

where the system matrices are $(A_i, B, C, D)$, $i = 0,\dots,m$ are real-valued and the time delays, $(\tau_0,\dots,\tau_m)$, are real numbers.

The following theorem is used to compute the $\mathscr{H}_\infty$ norm of a transfer function in the finite dimensional case.

**Theorem 1.1** *[1] Let $\xi > 0$ be such that the matrix $R = \xi^2 I - D^T D$ is non-singular. For $\omega \geq 0$, the matrix $G_o(j\omega) = C(j\omega I - A)^{-1}B + D$ has a singular value equal to $\xi$ if and only if $\lambda = j\omega$ is an eigenvalue of the Hamiltonian matrix*

$$H_\xi = \begin{bmatrix} A + BR^{-1}D^T C & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^T C) \end{bmatrix}.$$

Hence the $\mathscr{H}_\infty$ norm of $G$ satisfies

$$\|G_o\|_\infty = \sup\{\xi > 0 | H_\xi \text{ has an eigenvalue on the imaginary axis}\}. \quad (2)$$

This relation lays the basis of the well-established level set methods for computing $\mathscr{H}_\infty$ norm of finite dimensional systems (see, e.g. [2], for a quadratically converging algorithm).

In this talk, we extend the computation of $\mathscr{H}_\infty$ norm to the time-delay systems with the transfer function representation (1). The relation between the singular value of the transfer function and the corresponding Hamiltonian matrix remains valid. More precisely, let $\xi > 0$ be such that the matrix

$$D_\xi := D^T D - \xi^2 I$$

is non-singular. For $\omega \geq 0$, the matrix $G(j\omega)$ has a singular value equal to $\xi$ if and only if $\lambda = j\omega$ is a solution of the equation

$$\det H_\xi(\lambda) = 0, \quad (3)$$

where

$$\begin{aligned} H_\xi(\lambda) := \lambda I - M_0 \quad &- \sum_{i=1}^{m}\left(M_i e^{-\lambda\tau_i} + M_{-i} e^{\lambda\tau_i}\right) \\ &- \left(N_1 e^{-\lambda\tau_0} + N_{-1} e^{\lambda\tau_0}\right) \end{aligned}$$

and $M_0$, $N_1$, $N_{-1}$, $M_i$, $M_{-i}$ $i = 1,\dots,m$ depends on $\xi$ and the system matrices in (1).

We show that the nonlinear eigenvalue problem (3) is equivalent to a linear eigenvalue problem of the infinite dimensional Hamiltonian operator $\mathscr{L}_\xi$ on $X := \mathscr{C}([-\tau_{\max},\ \tau_{\max}], \mathbb{C}^{2n})$ which is defined by

$$\begin{aligned} \mathscr{D}(\mathscr{L}_\xi) &= \Big\{\phi \in X :\ \phi' \in X,\ \phi'(0) = M_0\phi(0) + \\ &\quad \sum_{i=1}^{m}(M_i\phi(-\tau_i) + M_{-i}\phi(\tau_i)) + N_1\phi(-\tau_0) + N_{-1}\phi(\tau_0)\Big\}, \\ \mathscr{L}_\xi\ \phi &= \phi'. \end{aligned}$$

Our approach to compute $\|G\|_\infty$ consists of two steps. In the first step inspired by (2), we compute using the method presented in [2],

$$\max\{\xi > 0 | \mathscr{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$$

where $\mathscr{L}_\xi^N$ is a matrix approximating $\mathscr{L}_\xi$. This problem can be interpreted as computing the $\mathscr{H}_\infty$ norm of an approximation of $G$ under mild conditions.

In the second step, the approximated results are corrected using Newton iteration on a set of equations which are obtained from the nonlinear eigenvalue problem (3) and characterize the peaks in the singular value plot.



### References

[1] S. Boyd, K. Balakrishnan, and P. Kabamba, "A bisection method for computing the $\mathscr{H}_\infty$ of a transfer matrix and related problems," Math Control Signals and Systems, 2(3), pp. 207-219, 1989.

[2] O. Bruinsma, and M. Steinbuch, "A fast algorithm to compute the $\mathscr{H}_\infty$ norm of a transfer function matrix," Systems and Control Letters, vol. 14, pp. 287-293, 1990.