Fixed-Order $\mathcal{H}^\infty$ Controller Design via HIFOO, a Specialized Nonsmooth Optimization Package

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***Abstract*— We report on our experience with fixed-order $\mathcal{H}^\infty$ controller design using the HIFOO toolbox. We applied HIFOO to various benchmark fixed (or reduced) order $\mathcal{H}^\infty$ controller design problems in the literature, comparing the results with those published for other methods. The results show that HIFOO can be used as an effective alternative to existing methods for fixed-order $\mathcal{H}^\infty$ controller design.**

I. Introduction

In this note, we report on our experience applying HIFOO [6] ($\mathcal{H}^\infty$ Fixed-Order Optimization) to various benchmarks for fixed-order $\mathcal{H}^\infty$ controller design. The plants in the examples are all finite-dimensional, linear time-invariant and multi-input-multi-output (MIMO). The controller order is fixed *a priori* to be less than than the order of plant. The design problem is to minimize the $\mathcal{H}^\infty$ norm of the transfer function for the closed loop plant. This is a difficult optimization problem due to the nonconvexity and nonsmoothness of the objective function. HIFOO uses a hybrid algorithm for nonsmooth, nonconvex optimization based on several techniques to attempt to find fixed-order controllers achieving the minimum closed-loop $\mathcal{H}^\infty$ norm. The results are compared with published results using other techniques.

Benchmark examples are chosen from both applied and academic test problems:

1) **AC**8**:** A 9^{th}-order state-space model of the linearized vertical plane dynamics of an aircraft [15];
2) **HE**1**:** A 4^{th}-order model of the longitudinal motion of a VTOL helicopter for typical loading and flight condition at the speed of 135 knots [22], and **VTOL**, a variation of this model;
3) **REA**2**:** A 4^{th}-order chemical reactor model [20], and **CR**, a variation of this model;
4) **AC**10**:** A 55^{th}-order aeroelastic model of a modified Boeing B-767 airplane at a flutter condition [9];
5) **BDT**2**:** An 82^{nd}-order realistic model of a binary distillation tower with pressure variation considered in model description [27];
6) **HF**1**:** A 130^{th}-order one-dimensional model for heat flow in a thin rod [19];
7) **CM**4**:** A 240^{th}-order cable mass model describing a hybrid parameter system representing nonlinear dynamic response of a relief valve used to protect a pneumatic system from overpressure [26];
8) **PA:** A 5^{th}-order model of a piezoelectric bimorph actuator system [8];
9) **HIMAT:** A 20^{th}-order model of an experimental highly maneuverable (HIMAT) airplane which is a scaled and remotely piloted version of an advanced fighter [17];
10) **VSC:** A 4^{th}-order quarter-car model consisting of one-fourth of the body mass and suspension components of a car and one wheel. This model is used extensively in the literature and captures many essential characteristics of a real suspension system;
11) **AUV:** This linear model of a cruise control system is obtained by linearizing the non-linear model of an autonomous underwater vehicle, Subzero III, around its cruising condition. Three SISO autopilots (speed, heading and depth autopilots) need to be developed for the flight control of Subzero III. The plant models for speed, heading and depth autopilots are 3^{rd}, 5^{th} and 6^{th}-order respectively [14];
12) **Enns' Example:** This 8^{th}-order plant was proposed by D. F. Enns [13]. This example is used as an academic test problem in the literature for designing reduced-order $\mathcal{H}^\infty$ controllers;
13) **Wang's Example:** This 4^{th}-order plant was suggested by J.-Z. Wang as a theoretical benchmark problem in [29], Example 6.2.

Note that benchmark examples $1-11$ are taken from real applications and $12-13$ are academic test problems. The problem data for examples $1-8$ are obtained from the COMPL$_e$IB library [23] and those for examples $9-13$ are collected from various papers in the literature. For another collection of results using HIFOO, see [18].

The rest of the paper is organized as follows. The problem of fixed-order $\mathcal{H}^\infty$ controller design is described and the optimization method used by HIFOO is summarized in Section II. Our computational results and comparisons with those published for other methods are given in Section III. Concluding remarks are in Section IV.

II. Problem Formulation and Optimization Method

Consider the standard feedback system with generalized plant, G, with state space realization

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \tag{II.1}$$

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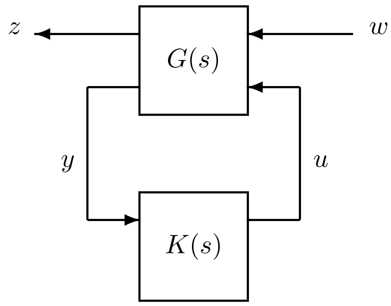

Fig. 1. Standard Feedback System

where $A \in \mathcal{R}^{n\times n}$, $D_{12} \in \mathcal{R}^{p_1\times m_2}$, $D_{21} \in \mathcal{R}^{p_2\times m_1}$ and other matrices have compatible dimensions. Let the controller have state space realization $K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$, where $A_K \in \mathcal{R}^{n_K\times n_K}$ and B_K, C_K, D_K have dimensions that are compatible with A_K and the plant matrices. The transfer function from the input w to output z is

$$\mathcal{F}_l(G,K) = G_{11} + G_{12}(I - G_{22}K)^{-1}G_{21} \quad \text{(II.2)}$$

where $G_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij}$ for $i,j = 1,2$.

The optimal $\mathcal{H}^\infty$ controller design can be formulated as minimization of the closed loop $\mathcal{H}^\infty$ norm function

$$\inf_{K \text{ stabilizing}} ||\mathcal{F}_l(G,K)||_\infty, \quad \text{(II.3)}$$

where K internally stabilizes the closed-loop system. When the controller order n_K equals the plant order n, methods are known to compute the controller that minimizes the $\mathcal{H}^\infty$ norm. However, unless n is very small, implementation of full-order controllers is generally not practical or desirable.

In order to circumvent this difficulty, we consider the same problem with the controller order n_K fixed to a number smaller than n. We refer to this as the *Fixed-Order $\mathcal{H}^\infty$ Controller Design* problem. The closed-loop $\mathcal{H}^\infty$ norm function is, in general, nonconvex and nonsmooth, and often is not differentiable at local minimizers. Thus, no method is known for finding a guaranteed global minimum. HIFOO attempts to locally minimize the objective function using a hybrid algorithm that includes the following elements:

- a quasi-Newton algorithm (BFGS) initial phase provides a fast way to approximate a local minimizer;
- a local bundle phase attempts to verify local optimality for the best point found by BFGS, and if this does not succeed,
- a gradient sampling phase [7], [5] attempts to refine the approximation of the local minimizer, returning a rough optimality measure.

All three of these optimization techniques use gradients which are automatically computed by HIFOO. No effort is made to identify the exceptional points where the gradients fail to exist. The algorithms are not defeated by the discontinuities in the gradients at exceptional points. The BFGS phase builds a highly ill-conditioned Hessian approximation matrix, and the bundle and gradient sampling final phases search for a point in parameter space for which a convex combination of gradients at nearby points has small norm. More details are given in [6]. We used HIFOO 1.5 [25], which differs from HIFOO 1.0 [6] in that, in HIFOO 1.5, the D_{22} block is allowed to be nonzero and specification of a sparsity pattern for the controller is possible. However, we did not make use of these features; D_{22} is zero for all the examples below.

Because HIFOO uses randomized starting points, and also the gradient sampling phase involves randomization, the same results are not obtained every time HIFOO is run. The results shown below are typical results sometimes obtained after several runs.

We did not attempt to compare running times of different methods. In our view, one of the biggest advantages of HIFOO is its ease of use. As one example, its running time can be controlled by providing a value for `options.cpumax`.

III. Results on Benchmark Problems

A. Examples from the COMPL$_e$IB Library

In [1], nonsmooth $\mathcal{H}^\infty$ synthesis algorithms are described and tested on various synthesis problem from the COMPL$_e$IB library [23]. The philosophy of using direct nonsmooth optimization is similar to ours but the algorithmic details are different. Fixed-order $\mathcal{H}^\infty$ controllers are designed for the problems and the performance of the nonsmooth $\mathcal{H}^\infty$ algorithm is compared with a specialized augmented Lagrangian algorithm [4], the Frank-Wolfe algorithm [12] and full-order $\mathcal{H}^\infty$ controller design method by the DGKF technique [10].

In the results given in [1], the nonsmooth $\mathcal{H}^\infty$ algorithm performs best for all benchmark problems except the plant REA2 for which the augmented Lagrangian algorithm gives a better result. We applied HIFOO to the same benchmark examples and compared our results with the augmented Lagrangian result for plant REA2 and the nonsmooth $\mathcal{H}^\infty$ results for the other examples. The results are given in Table I. The third and fourth columns display the final value of the $\mathcal{H}^\infty$ norm for the closed-loop plant along with the controller order, comparing the results from [1] with the results using HIFOO. For comparison, the second column shows the $\mathcal{H}^\infty$ norm for the closed-loop system using an optimal full-order controller.

As seen in Table I, HIFOO gives better performance than other algorithms for plants REA2 and BDT2 and the same performance for plants AC8, HE1, HF1 and CM4. Using its default randomly generated starting conditions, HIFOO has difficulty finding a stabilizing controller for AC10, because of the very different scalings of the variables. Therefore, we provided an initial stable starting point from [7].

B. Comparison with $\mathcal{H}^\infty$ Multidirectional Search Method

We consider static output-feedback $\mathcal{H}^\infty$ synthesis for the plants VTOL Helicopter (VTOL), Chemical Reactor (CR) and Piezoelectric Actuator (PA). The first two are slight

TABLE I

COMPARISON ON EXAMPLES FROM THE COMPL$_e$IB LIBRARY

Plant	($\|\mathcal{F}_l(G,K)\|_\infty, n_K$)		
	Full-Order	[1]	HIFOO
AC8	$(1.892, 9)$	$(2.005, 0)$	$(2.005, 0)$
HE1	$(0.0737, 4)$	$(0.154, 0)$	$(0.154, 0)$
REA2	$(1.135, 4)$	$(1.155^\dagger, 0)$	$(1.149, 0)$
AC10	$(3.23, 55)$	$(13.11, 0)$	$(13.790^*, 0)$
AC10	$(3.23, 55)$	$(10.21, 1)$	$(10.338^*, 1)$
BDT2	$(0.234, 82)$	$(0.8364, 0)$	$(0.6515, 0)$
HF1	$(0.447, 130)$	$(0.447, 0)$	$(0.447, 0)$
CM4	$(0.816, 240)$	$(0.816, 0)$	$(0.816, 0)$

† Augmented Lagrangian method * Stable Starting Point

variations on HE1 and REA2, respectively. The state-space data for these examples are taken from [3] to use the same data set as [2].

An algorithm combining multidirectional search (MDS) with nonsmooth optimization techniques is given in [2]. The algorithm is applied to the plants above for static output-feedback $\mathcal{H}^\infty$ synthesis and its results compared with the Augmented Lagrangian method (AL) described in [3]. We applied HIFOO to the same problems and the results are given in Table II.

TABLE II

COMPARISON WITH MULTIDIRECTIONAL SEARCH METHOD

Plant	($\|\mathcal{F}_l(G,K)\|_\infty, n_K$)		
	Full-Order	[2]	HIFOO
VTOL	$(0.0737, 4)$	$(0.157, 0)^\dagger$	$(0.154, 0)$
CR	$(1.135, 4)$	$(1.183, 0)$	$(1.168, 0)$
PA	numerically ill-posed	$(1.76e{-}4, 0)$	$(1.18e{-}4, 0)$

† Augmented Lagrangian method

The controllers obtained by HIFOO for static-output feedback $\mathcal{H}^\infty$ synthesis have lower closed-loop $\mathcal{H}^\infty$ cost compared to other methods for the benchmark problems above.

C. Enns' Benchmark Problem

We consider fixed-order $\mathcal{H}^\infty$ controller design of a plant proposed by Enns [13]. This example is used as a benchmark problem in the literature to design reduced-order $\mathcal{H}^\infty$ controllers. The optimal $\mathcal{H}^\infty$ norm achieved in closed-loop by a full-order (order 8) controller is 1.1272.

In [32], several controller reduction methods are compared, including weighted additive and coprime factor controller reduction methods, and these are applied to Enns' benchmark problem. In [21] and [31] reduced-order controllers are obtained by weighted $\mathcal{H}^\infty$ model reduction and a block-balanced truncating algorithm respectively. Recent enhancements of several frequency-weighted balancing related controller reduction methods are discussed in [28].

We applied HIFOO to the same benchmark example and compare the results with those obtained in [32] as well as by the other methods [21], [31], [28] in Table III. For all of orders 1 through 7, HIFOO finds controllers with lower closed-loop $\mathcal{H}^\infty$ norm. Therefore, the performance of HIFOO is better than other methods for this particular benchmark problem. Note that while the other methods compute a full-order controller first and then apply techniques to reduce its order, HIFOO does not compute a full-order controller, but computes low-order controllers directly.

TABLE III

COMPARISON ON ENNS' EXAMPLE

n_K	$\|\mathcal{F}_l(G,K)\|_\infty$				
	[32]	[21]	[31]	[28]	HIFOO
7	1.1960	1.1957	1.198	1.1950	1.1655
6	1.1960	1.1971	1.196	1.1960	1.1447
5	1.1950	1.1970	1.204	1.1960	1.1508
4	1.1950	1.1991	1.197	1.1960	1.1923
3	1.4880	1.8801	3.906	2.7580	1.1921
2	1.4150	1.9681	1.954	1.4130	1.2438
1	2.4670	73.2860	Unstable	Unstable	1.4256

D. HIMAT Example

Longitudinal dynamics of an experimental highly maneuverable (HIMAT) airplane make a well-known benchmark example for reduced-order robust controller design [17], [30]. The generalized plant has 20 states and the optimal $\mathcal{H}^\infty$ norm achieved in closed-loop by a full-order controller is 0.9708.

The controller reduction techniques in [17], [30] use frequency-weighted model reduction preserving $\mathcal{H}^\infty$ performance. We applied HIFOO to the HIMAT example as an alternative to controller reduction. The results can be seen in Table IV.

TABLE IV

COMPARISON ON HIMAT EXAMPLE

n_K	$\|\mathcal{F}_l(G,K)\|_\infty$		
	[17]	[30]	HIFOO
16	0.98	0.97	1.01
15	–	0.97	1.01
14	–	0.97	1.01
13	0.98	0.98	1.01
12	–	0.98	1.01
11	–	0.99	1.02
10	2.02	1.27	1.03
7	1.27	1.22	1.06
6	–	1.22	1.07

Note that HIFOO gives better performance compared to other methods when the controller order is low. When the controller order is close to the plant order, other methods perform better. However, the difference between performance is small. This example shows that although HIFOO gives good results when controller order is high, its best results are obtained when the controller order is small which is the case in almost all practical implementations.

E. Vehicle Suspension Control (VSC)

A simple quarter-car suspension model consists of one-fourth of the body mass and suspension components and one wheel. The model has 4 states and captures essential characteristics of a real suspension system. The suspension system is controlled by a hydraulic actuator for ride comfort, road holding ability and suspension deflection. An $\mathcal{H}^\infty$ control problem is formulated by weighting three different objectives for vehicle suspension [24].

In [11], a static output feedback $\mathcal{H}^\infty$ controller for the quarter-car suspension model with semi-active damper is obtained using a genetic algorithm. Table V shows the comparison between [11] and HIFOO. Note that HIFOO finds a static $\mathcal{H}^\infty$ controller achieving closed-loop $\mathcal{H}^\infty$ norm close to the optimal value for a fourth-order controller.

TABLE V
COMPARISON ON VEHICLE SUSPENSION CONTROL EXAMPLE

	$\|\mathcal{F}_l(G,K)\|_\infty$		
Plant	Full-Order	[11]	HIFOO
quarter-car suspension model with semi-active damper	3.216	7.640	3.975

F. Autonomous Underwater Vehicle (AUV)

In [14], autopilots (forward speed, heading and depth) are designed to control an autonomous underwater vehicle with performance objectives. It is desirable to have a low-order autopilot for implementation purposes. Therefore, a reduced-order $\mathcal{H}^\infty$ control problem is posed as a rank minimization problem and a solution is approximated by a trace minimization approach.

Table VI shows that HIFOO achieves lower closed-loop $\mathcal{H}^\infty$ norm with a smaller controller order compared to [14].

TABLE VI
COMPARISON ON AUTONOMOUS UNDERWATER VEHICLE EXAMPLE

	$(\|\mathcal{F}_l(G,K)\|_\infty, n_K)$		
Autopilots	Full-Order	[14]	HIFOO
Speed	$(0.9538, 3)$	$(0.9550, 1)$	$(0.9543, 1)$
Heading	$(0.9536, 5)$	$(0.9633, 3)$	$(0.9540, 2)$ $(0.9545, 1)$ $(0.9548, 0)$
Depth	$(0.9556, 6)$	$(0.9798, 3)$	$(0.9621, 1)$

G. Wang's Example

We consider the theoretical example in [29], Example 6.2. Controller approximation approaches preserving $\mathcal{H}^\infty$ performance are suggested in [17]. The $\mathcal{H}^\infty$ controller reduction problem is converted to a frequency weighted model reduction problem. The controller reduction method in [17] is generalized in [29].

In [16], algorithms based on a cone complementarity linearization idea are proposed to solve the nonconvex feasibility problems for controller order reduction. The results are compared with [29] and better performance is observed. We applied HIFOO to the same problem and the results are shown in Table VII. The closed-loop $\mathcal{H}^\infty$ norms for [29] and [16] are computed using the controllers shown in the corresponding papers and are less than the theoretical upper bounds in the papers. Note that the controllers found by HIFOO give closed-loop $\mathcal{H}^\infty$ norm close to the result for a full-order controller.

TABLE VII
COMPARISON ON WANG'S EXAMPLE

$(\|\mathcal{F}_l(G,K)\|_\infty, n_K)$			
Full-Order	[29]	[16]	HIFOO
$(50.640, 4)$	$(55.621, 3)$ $(55.639, 2)$	$(58.096, 3)$ $(55.624, 2)$	$(50.642, 2)$ $(50.645, 1)$ $(50.879, 0)$

IV. CONCLUDING REMARKS

In this note, we reported on results of applying the HIFOO Toolbox to various benchmark problems for fixed-order and reduced-order $\mathcal{H}^\infty$ design. The examples were mostly chosen from various applications and also included two academic test problems.

The performance of HIFOO is better compared to existing results in the literature in most cases. We conclude that HIFOO can be used a effective alternative method for fixed-order $\mathcal{H}^\infty$ controller design. HIFOO, which is written in MATLAB, is easy to use and is freely available on the web[1].

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