Design of Fixed-Order Robust Controllers via Nonsmooth Optimization

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Problem Definition

✔ The state-space equations of a generalized plant G are

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t)\\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t)\\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t)\end{aligned}$$

and the state-space realization for the controller K is

$$\begin{aligned}\dot{x}_K(t) &= A_K x_K(t) + B_K y(t)\\ u(t) &= C_K x_K(t) + D_K y(t)\end{aligned}$$

where $A \in \mathcal{R}^{n\times n}$, and $A_K \in \mathcal{R}^{n_K\times n_K}$. The controller order n_K is a design parameter.

Problem Definition (Cont'd)

✔ Let's connect the generalized plant and the controller,

$$\begin{aligned} \dot{x}_{cl}(t) &= A_{cl}x_{cl}(t) + B_{cl}w(t) \\ z(t) &= C_{cl}x_{cl}(t) + D_{cl}w(t) \end{aligned}$$

The closed-loop matrices contain the controller matrices as design parameters

Problem Definition (Cont'd)

✔ **Design constraints:** The controller is,

1) fixed order n_K
2) stabilizing closed-loop, $\alpha(A_{cl}) < 0$
3) minimizing H-infinity norm from w to z
   $\|T_{zw}\|_\infty = \sup_{w \in \mathbb{R}} \sigma(C_{cl}(jwI - A_{cl})^{-1}B_{cl} + D_{cl})$
4) itself stable, $\alpha(A_K) < 0$

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✔ We would like to solve the following problems

Fixed order H-infinity controller design $(1-3)$

Stable fixed order H-infinity controller design $(1-4)$

✔ Why are fixed-order controllers important?

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  - ✘ There is no known method for optimal H-infinity controller design if the controller order is less than plant order,
  - ✘ Low order and stable controller requirements are often conflicting

Optimization Problem

✔ HIFOO uses two phases: stabilization and performance optimization.

- ✘ stabilization phase: minimize $\max(\alpha(A_{CL}, \epsilon\, \alpha(A_K))$ where $\epsilon > 0$, quitting when stabilization is achieved
- ✘ performance phase: look for a local minimizer of

$$f(K) = \begin{cases} \infty \text{ if } \max(\alpha(A_{CL}), \alpha(A_K)) \geq 0 \\ \max(\|T_{zw}\|_\infty, \epsilon\|K\|_\infty) \text{ otherwise,} \end{cases}$$

where $\|K\|_\infty = \sup_{w\in\mathbb{R}} \sigma(C_K(jwI - A_K)^{-1}B_K + D_K)$.

Optimization Method

- ✔ HIFOO implements a hybrid algorithm for nonsmooth, nonconvex optimization, based on the following elements:
  - ✘ a quasi-Newton algorithm (BFGS) provides a fast way to approximate a local minimizer;
  - ✘ a local bundle method attempts to verify local optimality for the best point found by BFGS, and if this does not succeed,
  - ✘ gradient sampling attempts to refine the approximation of the local minimizer, returning a rough optimality measure.
- ✔ HIFOO uses randomized starting points,
- ✔ HIFOO accepts an option to control the running time

Optimization Method

✔ More information on optimization and HIFOO

- ✘ J. V. Burke, A. S. Lewis and M. L. Overton, "A robust gradient sampling algorithm for nonsmooth nonconvex optimization," *SIAM Journal on Optimization*, vol.15, pp. 751–779, 2005.
- ✘ J. V. Burke and M. L. Overton, "Variational Analysis of Non-Lipschitz Spectral Functions," *Mathematical Programming*, vol.90, pp. 317-352, 2001.
- ✘ M. Millstone, "HIFOO 1.5: Structured control of linear systems with a non-trivial feedthrough", M.S. thesis, Courant Institute of Mathematical Sciences, New York University, 2006.
- ✘ http://www.cs.nyu.edu/overton/software/hifoo/

Benchmarking

- ✔ Benchmark results for **Fixed-order H-infinity controller design**
  - ✘ 13 benchmark plants from industrial and academic problems
  - ✘ Results are compared with those published in the literature
  - ✘ Compleib library, Enns' Example, HIMAT Example and more...

Benchmarking - Comparison on Compleib Library

✔ HIFOO is effective for high order plants

Plant	$(\|\mathcal{F}_l(G,K)\|_\infty, n_K)$		
	Full-Order	Nonsmooth Hinf [Apkarian, Noll TAC06]	HIFOO
AC8	$(1.892, 9)$	$(2.005, 0)$	$(2.005, 0)$
HE1	$(0.0737, 4)$	$(0.154, 0)$	$(0.154, 0)$
REA2	$(1.135, 4)$	$(1.155^\dagger, 0)$	$(1.149, 0)$
AC10	$(3.23, 55)$	$(13.11, 0)$	$(12.83^*, 0)$
AC10	$(3.23, 55)$	$(10.21, 1)$	$(10.338^*, 1)$
BDT2	$(0.234, 82)$	$(0.8364, 0)$	$(0.6515, 0)$
HF1	$(0.447, 130)$	$(0.447, 0)$	$(0.447, 0)$
CM4	$(0.816, 240)$	$(0.816, 0)$	$(0.816, 0)$

Benchmarking - Comparison with Multidirectional Search Method[Apkarian, Noll 2006]

- ✔ HIFOO has slightly better performance
- ✔ HIFOO has good performance for numerically ill-posed problems

Plant	$(\|\mathcal{F}_l(G,K)\|_\infty, n_K)$		
	Full-Order	Multidirectional	HIFOO
VTOL	$(0.0737, 4)$	$(0.157, 0)^\dagger$	$(0.154, 0)$
CR	$(1.135, 4)$	$(1.183, 0)$	$(1.168, 0)$
PA	numerically ill-posed	$(1.76e{-}4, 0)$	$(1.18e{-}4, 0)$

Benchmarking - Comparison on Enns' Example

- ✔ The order of the plant is 8
- ✔ The achievable minimum H-infinity closed-loop norm using a full-order controller is 1.1272

	$\|\mathcal{F}_l(G,K)\|_\infty$				
n_K	Zhou	Kavranoglu	Wang	Varga	HIFOO
7	1.1960	1.1957	1.198	1.1950	1.1655
6	1.1960	1.1971	1.196	1.1960	1.1447
5	1.1950	1.1970	1.204	1.1960	1.1508
4	1.1950	1.1991	1.197	1.1960	1.1923
3	1.4880	1.8801	3.906	2.7580	1.1921
2	1.4150	1.9681	1.954	1.4130	1.2438
1	2.4670	73.2860	Unstable	Unstable	1.4256

Benchmarking - Comparison on HIMAT Example

n_K	$\|\mathcal{F}_l(G,K)\|_\infty$		
	Goddard	Wang	HIFOO
16	0.98	0.97	1.01
15	–	0.97	1.01
14	–	0.97	1.01
13	0.98	0.98	1.01
12	–	0.98	1.01
11	–	0.99	1.02
10	2.02	1.27	1.03
7	1.27	1.22	1.06
6	–	1.22	1.07

- ✔ The standard benchmark example for reduced-order controller design
- ✔ The order of the plant is 20
- ✔ The achievable minimum H-infinity closed-loop norm by a full-order controller is 0.9708

Remarks

- ✔ Low order controllers may achieve closed-loop H-infinity norm close to the optimal value achievable by a full-order controller
- ✔ HIFOO has good performance on high-order plants
- ✔ HIFOO performs well in numerically ill-posed problems
- ✔ HIFOO is a promising alternative method to controller reduction methods

Benchmarking

- ✔ Benchmark results for **Stable fixed order H-infinity controller design**
  - ✘ 17 benchmark plants from industrial and academic problems
  - ✘ Performances is compared to results published in the literature
  - ✘ Choi-Chung's Example, Four Disk System, Compleib library and more...

Benchmarking - Comparison on Choi-Chung's Example

n_K	γ_{n_K}	Methods	Controller Stability
16	25.430	Choi	Stable
12	21.787	Chou	Stable
8	43.167	Choi	Stable
8	37.551	Zeren	Stable
8	32.557	Gumussoy	Stable
8	24.790	Choi	Stable
4	12.015	**full**	Unstable
4	16.612	HIFOO	Stable
3	16.486	HIFOO	Stable
2	20.797	HIFOO	Stable
1	62.638	HIFOO	Stable

Benchmarking - Comparison on Four Disk System, $\beta = 10^{-1}$

n_K	γ_{n_K}	Methods	Controller Stability
24	0.237	Campos-Delgado	Stable
16	0.245	Zeren	Stable
16	0.241	Gumussoy	Stable
8	0.232	**full**	Unstable
8	0.235	HIFOO	Stable
7	0.236	HIFOO	Stable
6	0.236	HIFOO	Stable
5	0.235	HIFOO	Stable
4	0.274	HIFOO	Stable
3	0.307	HIFOO	Stable
2	0.347	HIFOO	Stable
1	0.649	HIFOO	Stable

Benchmarking - Compleib Library, High-Order Systems

AC10		BDT2		HF1		CM4	
n_K	γ_{n_K}	n_K	γ_{n_K}	n_K	γ_{n_K}	n_K	γ_{n_K}
55	0.633	82	0.234	130	0.447	240	0.816
8	8.984	8	0.531	8	0.447	8	0.824
7	9.003	7	0.542	7	0.447	7	0.819
6	9.376	6	0.534	6	0.447	6	0.818
5	9.570	5	0.559	5	0.447	5	0.817
4	9.869	4	0.604	4	0.447	4	0.818
3	9.869	3	0.578	3	0.447	3	0.817
2	9.869	2	0.576	2	0.447	2	0.817
1	10.863	1	0.643	1	0.447	1	0.817

Remarks

- ✔ HIFOO is often able to find stable controllers with lower order than given in the literature
- ✔ Since existing methods obtain controllers with order equal to or greater than the plant order, they are impractical in practice
- ✔ HIFOO finds stable low-order H-infinity controllers for various problems in the literature
- ✔ Existing stable H-infinity controller methods are conservative and HIFOO shows that there is a margin for improvement

Concluding Remarks

- ✔ HIFOO solves difficult problems in control theory and the results are not conservative compared to other existing methods in the literature
- ✔ The performance of HIFOO is very good for high order plants
- ✔ Detailed survey on HIFOO and control problems
  - ✘ S. Gumussoy and M. L. Overton, "Fixed-Order H-infinity Controller Design via HIFOO, a Specialized Nonsmooth Optimization Package," *ACC*, Seattle, 2008.
  - ✘ S. Gumussoy, M. Millstone and M. L. Overton, "$\mathcal{H}^\infty$ Strong Stabilization via HIFOO, a Package for Fixed-Order Controller Design," submitted to *CDC*, Cancun, 2008.
  - ✘ http://www.cs.nyu.edu/overton/software/hifoo/